

DISTRIBUTION OF LINEAR STATISTICS OF SINGULAR VALUES OF THE PRODUCT OF RANDOM MATRICES

F. GÖTZE, A. NAUMOV, AND A. TIKHOMIROV

ABSTRACT. In this paper we consider the product of two independent random matrices $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$. Assume that $X_{jk}^{(q)}, 1 \leq j, k \leq n, q = 1, 2$, are i.i.d. random variables with $\mathbb{E} X_{jk}^{(q)} = 0, \text{Var } X_{jk}^{(q)} = 1$. Denote by s_1, \dots, s_n the singular values of $\mathbf{W} := \frac{1}{n} \mathbf{X}^{(1)} \mathbf{X}^{(2)}$. We prove the central limit theorem for linear statistics of the squared singular values s_1^2, \dots, s_n^2 showing that the limiting variance depends on $\kappa_4 := \mathbb{E}(X_{11}^1)^4 - 3$.

One of the main questions studied in Random Matrix Theory (RMT) is the asymptotic analysis of spectra of random matrices when the dimension goes to infinity. For example it is well known since the pioneering work of Wigner [27] that the empirical spectral distribution function weakly converges to the semicircle law. Another well known case is the sample covariance matrices $\mathbf{W} = \mathbf{X}\mathbf{X}^T$, where \mathbf{X} is a matrix with independent entries, which was first studied in [20] by Marchenko and Pastur. The distribution of singular values of products of random matrices with independent entries has been intensively studied, see for example [4], [3] and [1].

All these results may be regarded as laws of large numbers for linear eigenvalue statistics. Thus fluctuations of such linear statistics of eigenvalues around its mean are of interest. There is a vast literature on this question. We mention the results of Jonsson [18], Bai and Silverman [7] Sinai and Soshnikov [24], Anderson and Zeitouni [5], Lytova and Pastur [19], Shcherbina [23], where the central limit theorem was proved. The aim of this paper is to investigate the case of singular values of *products* of random matrices with independent entries. It will be shown that in this case the central limit theorem holds as well and the limiting variance will be explicitly determined.

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1. INTRODUCTION

For any $m, n \geq 1$ we consider a family of independent real random variables $X_{j,k}^{(q)}$, $1 \leq j, k \leq n, q = 1, \dots, m$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Assume that the following conditions **(C0)** are fulfilled:

- a) $X_{jk}^{(q)}$ are identically distributed for $1 \leq j, k \leq n, q = 1, \dots, m$;
- b) for any $1 \leq j, k \leq n$

$$\mathbb{E} X_{jk}^{(q)} = 0 \text{ and } \mathbb{E}(X_{jk}^{(q)})^2 = 1;$$

- c) $\mathbb{E}(X_{jk}^{(q)})^4 = \mu_4 < \infty$.

The random variables $X_{jk}^{(q)}$ may depend on n , but for simplicity we shall not make this explicit in our notations.

We introduce m independent random matrices $\mathbf{X}^{(q)}, q = 1, \dots, m$, as follows

$$\mathbf{X}^{(q)} := \frac{1}{\sqrt{n}} [X_{jk}^{(q)}]_{j,k=1}^n.$$

Denote by s_1^2, \dots, s_n^2 the eigenvalues of the matrix $\mathbf{W}\mathbf{W}^T$, where $\mathbf{W} := \prod_{q=1}^m \mathbf{X}^{(q)}$ and define the empirical spectral measure by

$$F_n(x) = \frac{1}{n} \sum_{k=1}^n \mathbb{I}(s_k^2 \leq x).$$

Here and in what follows $\mathbb{I}\{B\}$ denotes the indicator of the event B .

A fundamental problem in the theory of random matrices is to determine the limiting distribution of F_n as the size of the random matrix tends to infinity. It was shown by N. Alexeev, F. Götze and A. Tikhomirov in [3] that there exists a function $G_m(x)$ such that

$$(1.1) \quad \lim_{n \rightarrow \infty} \sup_{x \in R} |\mathbb{E} F_n(x) - G_m(x)| = 0$$

and $G_m(x)$ are defined by its moments $M_k, k \in \mathbb{N}$,

$$M_k = \int_0^\infty x^k dG_m(x) = \frac{1}{mk+1} \binom{k}{mk+k}$$

which are so called Fuss-Catalan numbers. For $m = 1$ we get the well-known result of Marchenko-Pastur for sample covariance matrices. The Fuss-Catalan numbers satisfy the following simple recurrence relation

$$M_k = \sum_{k_0 + \dots + k_m = k-1} \prod_{\nu=0}^m M_{k_\nu}.$$

In [22] the density function $P_m(x)$ which satisfy

$$\int_0^{K_m} x^k P_m(x) dx = M_k$$

was found. Here $K_m := (m+1)^{m+1}/m^m$. An explicit formula for $P_m(x)$ is given in Appendix A.

Denote by $s(z)$ the Stieltjes transform of the distribution $G_m(x)$

$$s(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dG_m(x).$$

It can be shown that $s(z)$ satisfy the following equation

$$1 + zs(z) + (-1)^{m+1} z^m s(z)^{m+1} = 0.$$

The result (1.1) was proved under more general conditions than **(C0)**, it was assumed that the random variables may be non-identically distributed and satisfy a Lindeberg type condition for the second moments, for details see [3]. Under conditions **(C0)** the result (1.1) may be generalized and it can be shown that F_n weakly converges to G_m in probability. The latter may be rewritten in the following way

$$(1.2) \quad \int_{-\infty}^{\infty} f(\lambda) dF_n(\lambda) = \frac{1}{n} \sum_{k=1}^n f(s_k^2) \xrightarrow{p} \int_{-\infty}^{\infty} f(\lambda) dG_m(\lambda)$$

which is valid for all continuous and bounded real functions $f(\lambda)$. We may interpret (1.2) as the law of large numbers. The natural question is to investigate a fluctuation of linear statistic

$$S(\mathbf{W}) := \sum_{k=1}^n f(s_k^2)$$

around its mean.

1.1. Main result. Let $f(\lambda)$ be a smooth function with the Fourier transform given by

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) e^{-it\lambda} d\lambda,$$

We assume that $f(\lambda)$ satisfies the following condition

$$(1.3) \quad \int_{-\infty}^{\infty} (1 + |t|^5) |\hat{f}(t)| dt < \infty.$$

We will concentrate on the case of two random matrices, $m = 2$ and prove the following theorem which is the main result of this paper

Theorem 1.1. *Let $m = 2$. Under conditions **(C0)** and (1.3) the statistic*

$$S^{(0)} := S(\mathbf{W}) - \mathbb{E} S(\mathbf{W})$$

weakly converges to a Gaussian random variable G with zero mean and variance given by

$$(1.4) \quad \text{Var } G = \frac{\kappa_4}{2} \left[\int_{-a}^a f(\lambda^2) [p(\lambda) + \lambda p'(\lambda)] d\lambda \right]^2 \\ + \frac{1}{2\pi^2} \int_{-a}^a \int_{-a}^a \frac{(f(\lambda^2) - f(\mu^2))^2}{(\lambda - \mu)^2} \\ \times \frac{[p(\lambda) - p'(\lambda)(\lambda - \mu)]}{3p(\mu)} \frac{[4p_1(\mu)^4 + 11p_1(\mu)^2 + 4]}{4p_1(\mu)^2 + 3} d\lambda d\mu,$$

where $\kappa_4 = \mu_4 - 3$, $p_1(\lambda) = \pi p(\lambda)$, $p(\lambda) = |\lambda| P_2(\lambda^2)$ is the symmetrized Fuss-Catalan density, and $a = \sqrt{K_2}$.

Remark. Obviously the result of Theorem 1.1 depends on the distribution of $X_{ij}^{(q)}$, $1 \leq j, k \leq n, q = 1, 2$ in terms of the fourth cumulant rather than the second moment only. This means that the limiting behaviour is not universal in the usual sense, a fact which is typical for the central limit theorems of linear eigenvalue statistics.

Remark. The result of Theorem 1.1 may be extended on the case when $X_{ij}^{(q)}$, $1 \leq j, k \leq n, q = 1, 2$ are non-identically distributed. Here one has to impose additional assumptions, for example Lindeberg's condition on the tails of fourth moments of $X_{jk}^{(q)}$, see Section 3 for details.

Remark. The case $m > 2$ is much more difficult to analyse. One may derive a formula for $Y(x, t)$ (see the definition below). But it is not yet clear whether this expression is positive, due to the fact that the formula for $P_m(x)$, $m > 3$ is rather complicated. We plan to study this case in a subsequent paper.

1.2. Structure of the paper. We divide the proof of Theorem 1.1 into two parts. In the section 2 we consider the Gaussian case and derive an analogue of Theorem 1.1. Our method will be based on the result of Lytova and Pastur [19] and Tikhomirov [25], [26]. In the section 3 we investigate the difference between the general case and the Gaussian case. Here we will use the methods of Bentkus, see [8] and Tikhomirov, see [25], [26]. All auxiliary facts about Fuss-Catalan distribution, unitary matrix decomposition and its derivatives are collected in Appendix A-D.

1.3. Applications. One motivation for investigating the asymptotic distribution of products of random matrices follows from recent applications in wireless telecommunication, see [21]. Compare as well very recent results in [29]. Other immediate applications are in finance [10] and quantum entanglement [13], [30].

1.4. History. There are many papers on the CLT for linear eigenvalue statistics of random matrices. We mention the results of Jonsson [18], Bai and Silverman [7] Sinai and Soshnikov [24], Anderson and Zeitouni [5], Lytova and Pastur [19], Shcherbina [23]. In our setting the result for $m = 1$ was derived

by Lytova and Pastur in [19]. We will use their ideas in the proof of Theorem 1.1. One may also find a lot of information about the CLT for linear eigenvalues statistics in the book of Bai and Silverstein [6]. We also believe that one may apply the result of [7] together with [28] to derive Theorem 1.1 when restricting oneself to the class of *analytic* test functions with a different (implicit) representation for the variance using Cauchy integrals.

The distribution of singular values of products of random matrices with independent entries has been intensively studied in many papers. The relevant literature is much too extensive in order to describe it here in detail, see for example [4], [3], [22], [1], [15] and very recent result [14]. The central limit theorem for product of complex Ginibre matrices was derived by Breuer and Duits in [11]. It is known that in the complex Ginibre case the squares of singular values of \mathbf{W} form a determinantal point process and the joint density function is a bi-orthogonal ensemble, see [2].

1.5. Notations. In what follows we will use the following notations. Denote by $\|\mathbf{A}\|, \|\mathbf{A}\|_2$ the operator and Hilbert-Schmidt norms of \mathbf{A} respectively. As usual $\text{Tr } \mathbf{A} = \sum_{i=1}^n \mathbf{A}_{ii}$. We assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By $\text{Var}(\xi)$ we mean $\mathbb{E} \xi^2 - (\mathbb{E} \xi)^2$, where \mathbb{E} is the mathematical expectation with respect to \mathbb{P} . By C and c we denote some constants which do not depend on n . We introduce the symmetrized version of f , i.e.

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \geq 0, \\ f(-x) & \text{if } x < 0. \end{cases}$$

By $*$ we denote the convolution operation, i.e. $f * g(t) = \int_0^t f(s)g(t-s) ds$.

2. THE GAUSSIAN CASE

In this section we consider the special case when $X_{jk}^{(q)}, 1 \leq j, k \leq n, q = 1, 2$ has the Gaussian distribution. We change our notations of matrices and denote by $\mathbf{Y}^{(q)}, q = 1, 2$ the matrix $\mathbf{X}^{(q)}$ with $X_{jk}^{(q)}$ replaced by the Gaussian random variables. The main result of this section is the following theorem.

Theorem 2.1. *Let $\mathbf{Y}^{(q)} = n^{-1/2} \{Y_{jk}^{(q)}\}_{j,k=1}^n, q = 1, 2$, be independent random matrices such that the random variables $Y_{jk}^{(q)}, j, k = 1, \dots, n, q = 1, 2$, satisfy the conditions **(C0)**. Then the statistic*

$$S^{(0)} := S(\mathbf{W}) - \mathbb{E} S(\mathbf{W})$$

weakly converges to the Gaussian random variable G with zero mean and variance given by

$$(2.1) \quad \text{Var}[G] = \frac{1}{2\pi^2} \int_{-a}^a \int_{-a}^a \frac{(f(\lambda^2) - f(\mu^2))^2}{(\lambda - \mu)^2} \times \frac{[p(\lambda) - p'(\lambda)(\lambda - \mu)]}{3p(\mu)} \frac{[4p_1(\mu)^4 + 11p_1(\mu)^2 + 4]}{4p_1(\mu)^2 + 3} d\lambda d\mu,$$

where $p_1(\lambda) = \pi p(\lambda)$, $p(\lambda) = |\lambda|P_2(\lambda^2)$ is the symmetrized Fuss-Catalan density, and $a = \sqrt{K_2}$.

2.1. Symmetrization. Before we start to prove Theorem 2.1 we will introduce and prove a simple Lemma. Let ξ^2 be a positive random variable with the distribution function $F(x)$. Define $\tilde{\xi} := \varepsilon\xi$, where ε denotes a Rademacher random variable with $\mathbb{P}\{\varepsilon = \pm 1\} = 1/2$ which is independent of ξ . Let $\tilde{F}(x)$ denote the distribution function of $\tilde{\xi}$. It satisfies the following equation

$$(2.2) \quad \tilde{F}(x) = 1/2(1 + \text{sgn}\{x\} F(x^2)),$$

Lemma 2.2. For any one-sided distribution function $F(x)$ and $G(x)$ we have

$$\sup_{x \geq 0} |F(x) - G(x)| = 2 \sup_x |\tilde{F}(x) - \tilde{G}(x)|,$$

where $\tilde{F}(x)$ ($\tilde{G}(x)$) denotes the symmetrization of $F(x)$ ($G(x)$ respectively) according to (2.2).

Proof. By (2.2), we have for any $x \geq 0$

$$\begin{aligned} F(x) &= 2\tilde{F}(\sqrt{x}) - 1 \\ G(x) &= 2\tilde{G}(\sqrt{x}) - 1. \end{aligned}$$

This implies

$$\sup_{x \geq 0} |F(x) - G(x)| = 2 \sup_{x \geq 0} |\tilde{F}(\sqrt{x}) - \tilde{G}(\sqrt{x})| = 2 \sup_x |\tilde{F}(x) - \tilde{G}(x)|.$$

Thus Lemma is proved. \square

We apply this Lemma to the distribution of the squared singular values of the matrix \mathbf{W} . Let us denote

$$(2.3) \quad \mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{Y}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & \mathbf{Y}^{(m-\nu+1)T} \end{pmatrix} \text{ and } \mathbf{J} := \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}.$$

For any $1 \leq a, b \leq m$, put

$$(2.4) \quad \mathbf{V}_{[a,b]} = \begin{cases} \prod_{k=a}^b \mathbf{H}^{(k)}, & \text{for } a \leq b, \\ \mathbf{I} & \text{otherwise,} \end{cases}$$

and $\mathbf{V} = \mathbf{V}_{[1,m]}$, $\hat{\mathbf{V}} = \mathbf{V}\mathbf{J}$.

Note that $\hat{\mathbf{V}}$ is a symmetric matrix. The eigenvalues of the matrix $\hat{\mathbf{V}}$ are $-s_1, \dots, -s_n, s_n, \dots, s_1$. Note that the symmetrization of the distribution function $F_n(x)$ is a function $\tilde{F}_n(x)$ which is the empirical distribution function of the eigenvalues of the matrix $\hat{\mathbf{V}}$. According to Lemma 2.2 we get

$$\sup_x |\mathbb{E} F_n(x) - G_m(x)| = 2 \sup_x |\mathbb{E} \tilde{F}_n(x) - \tilde{G}_m(x)|,$$

and (1.2) may be rewritten as follows

$$\int_{-\infty}^{\infty} \tilde{f}(x) d\tilde{F}_n(x) = \frac{1}{2n} \sum_{k=1}^n [f(s_k) + f(-s_k)] \xrightarrow{p} \int_{-\infty}^{\infty} \tilde{f}(x) d\tilde{G}_m(x).$$

It is straightforward to check that the Stieltjes of $\tilde{G}(x)$ satisfies the following equation

$$(2.5) \quad 1 + zs(z) + (-1)^{m+1} z^{m-1} s^{m+1}(z) = 0.$$

To finish our linearization we mention that

$$\int_0^{\infty} f(x) dF_n(x) = \int_{-\infty}^{\infty} \tilde{f}(x^2) d\tilde{F}_n(x).$$

This means that we may substitute $f(x)$ by $f(x^2)$ and consider its symmetrization $\tilde{f}(x)$. In what follows we will consider symmetrized distribution functions only and omit the symbol "˜" in the corresponding notation. In the new notations we will have

$$S^0 = \frac{1}{2} [\text{Tr } f(\hat{\mathbf{V}}) - \mathbb{E} \text{Tr } f(\hat{\mathbf{V}})].$$

2.2. Empirical Poincaré Inequalities. Following [9] we say that a probability measure μ on \mathbb{R}^d satisfies a Poincaré-type inequality with constant σ^2 if for any bounded smooth function g on \mathbb{R}^d with gradient ∇g ,

$$(2.6) \quad \text{Var}(g) \leq \sigma^2 \int |\nabla g|^2 d\mu,$$

where $\text{Var}(g) = \int g^2 d\mu - (\int g d\mu)^2$. In this case we write $PI(\sigma^2)$ for short.

Assume that the random variables X_1, \dots, X_n have a joint distribution μ on \mathbb{R}^n , satisfying the Poicare-type inequality (2.6). Given a bounded smooth complex-valued function f on the real line, one may apply (2.6) to

$$g(x_1, \dots, x_n) = \frac{f(x_1) + \dots + f(x_n)}{n} = \int f(x) dF_n(x),$$

where F_n is the empirical measure, defined for observation $X_1 = x_1, \dots, X_n = x_n$. Since

$$|\nabla g|^2 = \frac{|f'(x_1)|^2 + \dots + |f'(x_n)|^2}{n^2} = \frac{1}{n} \int |f'(x)|^2 dF_n(x),$$

we obtain the following statement, see [9][Proposition 4.3],

Statement 2.3. Under $PI(\sigma^2)$, for any smooth F -integrable function $f : \mathbb{R} \rightarrow \mathbb{C}$,

$$\mathbb{E} \left| \int f(x) dF_n(x) - \int f(x) dF(x) \right|^2 \leq \frac{\sigma^2}{n} \int |f'(x)|^2 dF(x),$$

where $F(x) := \mathbb{E} F_n(x)$.

We will use the following linearization trick from [12]. Let us consider the matrix $\hat{\mathbf{V}} = [\prod_{j=1}^{m-1} \mathbf{H}^{(j)}] \mathbf{H}^{(m)} \mathbf{J}$. We form the following $mn \times mn$ matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{O} & \mathbf{H}^{(1)} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{H}^{(2)} & \mathbf{O} & \dots & \mathbf{O} \\ & & & & \dots & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{H}^{(m-1)} \\ \mathbf{H}^{(m)} \mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \end{bmatrix}$$

Then the m -th power of \mathbf{M} is a diagonal block matrix, there the first block is equal to $\hat{\mathbf{V}}$, the second $-\mathbf{H}^{(2)} \mathbf{H}^{(3)} \dots \mathbf{H}^{(m)} \mathbf{J} \mathbf{H}^{(1)}$ and so on. The eigenvalues of \mathbf{M}^m are the eigenvalues of $\hat{\mathbf{V}}$ with multiplicity m . We denote the eigenvalues of \mathbf{M} by $\lambda_1, \dots, \lambda_{mn}$ and their empirical distribution function by $G_n(\lambda)$. Then we have for an even function f

$$\int f(x) dF_n(x) = \frac{1}{n} \sum_{j=1}^n f(s_j) = \frac{1}{2nm} \sum_{j=1}^{2mn} f(\lambda_j^m) = \int f(\lambda^m) dG_n(\lambda).$$

Without loss of generality we assume that $\lambda_1, \dots, \lambda_n$ are real positive eigenvalues $s_1^{1/m}, \dots, s_n^{1/m}$. All other eigenvalues may be derived by a rotation on an angle $\theta_k = \frac{k\pi}{m}, k = 1, \dots, 2m-1$. Let $\theta_0 = 0$. We denote the empirical spectral distribution of $e^{i\theta_k} \lambda_1, \dots, e^{i\theta_k} \lambda_n$ by $G_{n,k}$. It is easy to see that

$$(2.7) \quad \int f(\lambda^m) dG_n(\lambda) = \frac{1}{2m} \sum_{k=0}^{2m-1} \int_{\mathbb{T}_k} f(\lambda^m) dG_{n,k}(\lambda),$$

where $\mathbb{T}_k = e^{i\theta_k} \mathbb{R}$.

The joint distribution P of the collection $\{Y_{jk}^{(q)}, j, k = 1, \dots, n, q = 1, \dots, m\}$ represents a product probability measure on the Euclidian space \mathbb{R}^N of dimension $N = mn^2$, while the joint distribution μ of the spectral values $\lambda_1, \dots, \lambda_n$ is a probability measure on \mathbb{R}^n , obtained from P as the image under the map $T = P \cdot S$, where S is the map from matrices to their eigenvalues and P is the projector on the subspace of the dimension n . We will apply the following Lemma (see [9][Lemma 7.1])

Lemma 2.4. Let μ_1, \dots, μ_N be probability measures on \mathbb{R} , satisfying $PI(\sigma^2)$. The image of the product measure $P = \mu_1 \otimes \dots \otimes \mu_N$ under any Lipshitz map $T : \mathbb{R}^N \rightarrow \mathbb{R}^n$ satisfies $PI(\sigma^2 \|T\|_{Lip}^2)$, where

$$\|g\|_{Lip} := \sup_{x \neq y} \frac{\rho_2(g(x), g(y))}{\rho_1(x, y)}$$

and ρ_1, ρ_2 are metrics in \mathbb{R}^N and \mathbb{R}^n respectively.

In our case

$$\sum_{j=1}^n |\lambda_j - \lambda'_j|^2 \leq \|\mathbf{M} - \mathbf{M}'\|_2^2 = \frac{2}{n} \sum_{q=1}^m \sum_{j,k=1}^n |Y_{jk}^{(q)} - (Y_{jk}^{(q)})'|^2$$

and $\|T\|_{Lip} = \frac{\sqrt{2}}{\sqrt{n}}$. Since $Y_{jk}^{(q)}$ satisfies $PI(\sigma^2)$ it follows from (2.7) and Statement (2.3) that

$$(2.8) \quad \mathbb{E} \left| \int f(x) dF_n(x) - \int f(x) dF(x) \right|^2 \leq \frac{\sigma^2 m^2}{n^2} \int |x|^{\frac{2m-2}{m}} |f'(x)|^2 dF(x).$$

2.3. Proof of CLT in the Gaussian case. In this subsection we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let us denote the characteristic function of S^0 by $Z_n(x)$, i.e.

$$Z_n(x) := \mathbb{E} e^{ixS^0}.$$

To prove Theorem 2.1 it is sufficient to derive that

$$\lim_{n \rightarrow \infty} Z_n(x) = Z(x),$$

where $Z(x)$ is a characteristic function of the Gaussian random variable G with zero mean and variance given by the formula (2.1), i.e.,

$$Z(x) = \mathbb{E} e^{itG} = e^{-\text{Var}[G]t^2/2}.$$

One has to show that $Z(x) = 1 - \text{Var}[G] \int_0^x y Z(y) dy$. Since

$$Z_n(x) = 1 + \int_0^x Z'_n(y) dy,$$

similarly to Lytova and Pastur ([19]) it is sufficient to prove that any converging subsequences $\{Z_{n_l}\}$ and $\{Z'_{n_l}\}$ satisfy

$$(2.9) \quad \lim_{n_l \rightarrow \infty} Z_{n_l}(x) = Z(x), \quad \lim_{n_l \rightarrow \infty} Z'_{n_l}(x) = -xZ(x) \text{Var}[G].$$

and show that $\text{Var}[G]$ is given by the formula (2.1).

Taking derivative of $Z_n(x)$ we get

$$Z'_n(x) = i \mathbb{E} S^0 e^{ixS^0} = \frac{i}{2} \int_{-\infty}^{\infty} \hat{f}(t) \mathbb{E} [\text{Tr } \mathbf{U}(t) - \mathbb{E} \text{Tr } \mathbf{U}(t)] e^{ixS^0} dt,$$

where we have applied the Fourier inverse formula

$$f(\lambda) = \int_{-\infty}^{\infty} \hat{f}(t) e^{it\lambda} dt.$$

We introduce further notations

$$\begin{aligned} u_n(t) &:= \frac{1}{2} \operatorname{Tr} \mathbf{U}(t), \\ u_n^0(t) &:= u_n(t) - \mathbb{E} u_n(t), \\ Y_n(x, t) &:= \mathbb{E} u_n^0(t) e^{ixS^0}. \end{aligned}$$

In these notations we may rewrite $Z'_n(x)$ as follows

$$(2.10) \quad Z'_n(x) = i \int_{-\infty}^{\infty} \hat{f}(t) Y_n(x, t) dt.$$

From unitary matrix representation (B.4) it follows that

$$u_n(t) = \sum_{j=1}^n U_{jj}(t) = \sum_{j=1}^n U_{j+n, j+n}(t).$$

The following Lemma gives the estimates for the variance of $u_n(t)$, its derivative $u'_n(t)$ with respect to the argument t , and $Y_n(x, t)$.

Lemma 2.5. *Under condition of Theorem 2.1 we have*

$$\operatorname{Var}(u_n(t)) \leq C_1 t^2, \quad \operatorname{Var}(u'_n(t)) \leq C_2(1 + t^2), \quad |Y_n(x, t)| \leq \sqrt{C_1} t.$$

Proof. The statement of this Lemma for $u_n(t)$ and $u'_n(t)$ follows from (2.8) applied to $f(x) = \cos(tx)$ and $f(x) = -x \sin(tx)$ respectively. From the Cauchy-Schwarz inequality we conclude that

$$|Y_n(x, t)| = |\mathbb{E}((u_n(t) - \mathbb{E} u_n(t)) e^{ixS^0})| \leq \operatorname{Var}^{1/2}(u_n(t)) \leq \sqrt{C_1} t.$$

□

From Lemma 2.5 we may conclude that

$$\left| \frac{\partial Y_n(x, t)}{\partial t} \right| \leq \operatorname{Var}^{1/2}(u'_n(t)) \leq C_2^{1/2} \sqrt{1 + t^2}$$

and

$$\left| \frac{\partial Y_n(x, t)}{\partial x} \right| \leq \operatorname{Var}^{1/2}(u_n(t)) \operatorname{Var}^{1/2}(S_0) \leq C_1^{1/2} t \sup_{\lambda \in \mathbb{R}} f'(\lambda)$$

One may see that $Y_n(x, t)$ is bounded and equicontinuous on any finite set of \mathbb{R}^2 . Similarly to Lytova and Pastur it is sufficient to show that any uniformly converging subsequence of $\{Y_n\}$ has the same limit Y , leading to (2.9).

2.4. Product of two random square matrices. Let $m = 2$. We investigate the quantity $Y_n(x, t)$. Applying Duhamel's formula

$$\mathbf{U}(t) = \mathbf{I} + i \int_0^t \hat{\mathbf{V}} \mathbf{U}(s) ds$$

we will have

$$Y_n(x, t) = \frac{i}{2} \int_0^t \mathbb{E} [\operatorname{Tr} \hat{\mathbf{V}} \mathbf{U}(s) - \mathbb{E} \operatorname{Tr} \hat{\mathbf{V}} \mathbf{U}(s)] e^{ixS^0} ds = \frac{1}{2} \mathcal{A}_1 + \frac{1}{2} \mathcal{A}_2,$$

where

$$\mathcal{A}_1 = \frac{i}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E}[Y_{jk}^{(1)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} - \mathbb{E} Y_{jk}^{(1)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j}] e^{ixS^0} ds$$

and

$$\begin{aligned} \mathcal{A}_2 &= \frac{i}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E}[Y_{jk}^{(2)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{j+n,k+n} \\ &\quad - \mathbb{E} Y_{jk}^{(2)} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{j+n,k+n}] e^{ixS^0} ds. \end{aligned}$$

Let us consider the term \mathcal{A}_1 . Applying $\mathbb{E} \xi f(\xi) = \mathbb{E} \xi^2 \mathbb{E} f'(\xi)$ valid for Gaussian random variable ξ with zero mean, we get

$$(2.11) \quad \mathcal{A}_1 = I_1 + I_2,$$

where

$$I_1 = \frac{i}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E} \left[\left[\mathbf{H}^{(2)} \mathbf{J} \frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(1)}} \right]_{k,j} - \mathbb{E} \left[\mathbf{H}^{(2)} \mathbf{J} \frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(1)}} \right]_{k,j} \right] e^{ixS^0} ds$$

and

$$I_2 = -\frac{x}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E}[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} \frac{\partial S}{\partial Y_{jk}^{(1)}} e^{ixS^0} ds.$$

From Lemma B.3 it follows that

$$\begin{aligned} &\sum_{j,k=1}^n \left[\mathbf{H}^{(2)} \mathbf{J} \frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(1)}} \right]_{k,j} \\ &= \frac{i}{\sqrt{n}} \sum_{j,k=1}^n \sum_{l=1}^{2n} [\mathbf{H}^{(2)} \mathbf{J}]_{k,l} [\mathbf{U} \mathbf{H}^{(1)}]_{l,k+n} * [\mathbf{U}]_{j,j}(s) \\ &\quad + \frac{i}{\sqrt{n}} \sum_{j,k=1}^n \sum_{l=1}^{2n} [\mathbf{H}^{(2)} \mathbf{J}]_{k,l} [\mathbf{U} \mathbf{H}^{(1)}]_{j,k+n} * [\mathbf{U}]_{l,j}(s) \\ &= \frac{i}{\sqrt{n}} \int_0^s u_n(s-s_1) \sum_{k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s_1) \mathbf{H}^{(1)}]_{k,k+n} ds_1 \\ &\quad + \frac{i}{\sqrt{n}} \int_0^s \sum_{j,k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s_1)]_{k,j} [\mathbf{U}(s-s_1) \mathbf{H}^{(1)}]_{j,k+n} ds_1. \end{aligned}$$

Let us denote

$$t_n(s) := \sum_{k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s) \mathbf{H}^{(1)}]_{k,k+n}.$$

In these notations we may write, applying Lemma C.1,

$$I_1 = -\frac{1}{n} \int_0^t ds \int_0^s \mathbb{E}[u_n(s-s_1) t_n(s_1) - \mathbb{E} u_n(s-s_1) t_n(s_1)] e^{ixS^0} ds_1 + r_n(t),$$

where

$$|r_n(t)| \leq C \frac{t^3}{\sqrt{n}}.$$

In what follows for simplicity we will not specify the term $r_n(t)$, but one should have in mind that $r_n(t)$ goes to zero as n goes to infinity. Let us rewrite the difference $\mathbb{E}[t_n(s_1)u_n(s-s_1) - \mathbb{E}t_n(s_1)u_n(s-s_1)]$. We have

$$(2.12) \quad \begin{aligned} t_n(s_1)u_n(s-s_1) &= t_n^0(s_1)u_n^0(s-s_1) + t_n^0(s_1)\mathbb{E}u_n(s-s_1) \\ &\quad + u_n^0(s-s_1)\mathbb{E}t_n(s_1) + \mathbb{E}t_n(s_1)\mathbb{E}u_n(s-s_1). \end{aligned}$$

and

$$(2.13) \quad \begin{aligned} t_n(s_1)u_n(s-s_1) - \mathbb{E}t_n(s_1)u_n(s-s_1) &= t_n^0(s_1)u_n^0(s-s_1) \\ &\quad + t_n^0(s_1)\mathbb{E}u_n(s-s_1) + u_n^0(s-s_1)\mathbb{E}t_n(s_1) \\ &\quad - \mathbb{E}t_n^0(s_1)u_n^0(s-s_1). \end{aligned}$$

From (2.13) the term I_1 may be rewritten as follows

$$\begin{aligned} I_1 &= -\frac{1}{n} \int_0^t ds \int_0^s \mathbb{E}u_n(s-s_1)\mathbb{E}t_n^0(s_1)e^{ixs_0} ds_1 \\ &\quad - \frac{1}{n} \int_0^t ds \int_0^s \mathbb{E}t_n(s-s_1)Y_n(x, s_1) ds_1 + r_n(t) =: I_{11} + I_{12} + r_n(t). \end{aligned}$$

Let us investigate $t_n(s)$. We may write

$$\begin{aligned} \mathbb{E}t_n(s) &= \frac{1}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E}Y_{jk}^{(2)}[\mathbf{U}(s)\mathbf{H}^{(1)}]_{k+n,j+n} \\ &= \frac{1}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} \left[\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(2)}} \mathbf{H}^{(1)} \right]_{k+n,j+n} + \frac{1}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} \left[\mathbf{U}(s) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{k+n,j+n}. \end{aligned}$$

From Lemma B.4 we conclude

$$\begin{aligned} (2.14) \quad &\sum_{j,k=1}^n \left[\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(2)}} \mathbf{H}^{(1)} \right]_{k+n,j+n} \\ &= \frac{i}{\sqrt{n}} \int_0^s \sum_{j,k=1}^n \sum_{l=1}^{2n} [\mathbf{U}(s_1)]_{k+n,k+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s-s_1)]_{j+n,l} [\mathbf{H}^{(1)}]_{l,j+n} ds_1 \\ &\quad + \frac{i}{\sqrt{n}} \int_0^s \sum_{j,k=1}^n \sum_{l=1}^{2n} [\mathbf{U}(s_1)]_{l,k+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s-s_1)]_{j+n,k+n} [\mathbf{H}^{(1)}]_{l,j+n} ds_1 \\ &= \frac{i}{\sqrt{n}} \int_0^s u_n(s_1) \sum_{j=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s-s_1) \mathbf{H}^{(1)}]_{j+n,j+n} ds_1 \\ &\quad + \frac{is}{\sqrt{n}} \sum_{k=1}^n [\mathbf{U}(s) \hat{\mathbf{V}}]_{k+n,k+n}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \sum_{j=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s - s_1) \mathbf{H}^{(1)}]_{j+n, j+n} &= \sum_{j=1}^n [\hat{\mathbf{V}} \mathbf{U}(s - s_1)]_{j+n, j+n} \\ &= -i \sum_{j=1}^n [\mathbf{U}'(s - s_1)]_{j+n, j+n} = -iu'_n(s - s_1), \end{aligned}$$

and

$$\sum_{k=1}^n [\mathbf{U}(s) \hat{\mathbf{V}}]_{k+n, k+n} = -iu'_n(s).$$

For the second term we have

$$\sum_{j,k=1}^n \mathbb{E} \left[\mathbf{U}(s) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{k+n, j+n} = \sqrt{n} \mathbb{E} u_n(s).$$

It follows that

$$\mathbb{E} t_n(s) = \frac{1}{n} \int_0^s \mathbb{E} u_n(s_1) u'_n(s - s_1) ds_1 + \mathbb{E} u_n(s) + \frac{s}{n} \mathbb{E} u'_n(s).$$

and

$$\begin{aligned} I_{12} &= -\frac{1}{n^2} \int_0^t ds \int_0^s Y_n(x, s_1) ds_1 \int_0^{s-s_1} \mathbb{E} u_n(s_2) u'_n(s - s_1 - s_2) ds_2 \\ &\quad - \frac{1}{n} \int_0^t ds \int_0^s Y_n(x, s_1) \mathbb{E} u_n(s - s_1) ds_1 \\ &\quad - \frac{1}{n^2} \int_0^t ds \int_0^s Y_n(x, s_1) \mathbb{E} u'_n(s - s_1) ds_1. \end{aligned}$$

Since $|Y_n(x, s)| \leq C$ (see Lemma 2.5) and $\mathbb{E} |u'_n(s - s_1)| \leq n\sqrt{n}$ we get

$$\begin{aligned} I_{12} &= -\frac{1}{n^2} \int_0^t ds \int_0^s Y_n(x, s_1) ds_1 \int_0^{s-s_1} \mathbb{E} u_n(s_2) u'_n(s - s_1 - s_2) ds_2 \\ &\quad - \frac{1}{n} \int_0^t ds \int_0^s Y_n(x, s_1) \mathbb{E} u_n(s - s_1) ds_1 + r_n(t) \end{aligned}$$

Applying Lemma 2.5 we may write

$$\begin{aligned} I_{12} &= -\frac{1}{n^2} \int_0^t ds \int_0^s Y_n(x, s_1) ds_1 \int_0^{s-s_1} \mathbb{E} u_n(s_2) \mathbb{E} u'_n(s - s_1 - s_2) ds_2 \\ &\quad - \frac{1}{n} \int_0^t ds \int_0^s Y_n(x, s_1) \mathbb{E} u_n(s - s_1) ds_1 + r_n(t). \end{aligned}$$

Changing the limits of integration we get

$$I_{12} = -\frac{1}{n^2} \int_0^t Y_n(x, s) ds \int_0^{t-s} \mathbb{E} u_n(s_1) \mathbb{E} u_n(t - s - s_1) ds_1 + r_n(t).$$

We investigate now $\mathbb{E} t_n^0(s) e^{ixS^0}$.

$$\begin{aligned}
& \mathbb{E} t_n^0(s) e^{ixS^0} \\
&= \frac{1}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} [Y_{jk}^{(2)} [\mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n,j+n} - \mathbb{E} Y_{jk}^{(2)} [\mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n,j+n}] e^{ixS^0} \\
&= \frac{1}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} \left[\left[\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(2)}} \mathbf{H}^{(1)} \right]_{k+n,j+n} - \mathbb{E} \left[\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(2)}} \mathbf{H}^{(1)} \right]_{k+n,j+n} \right] e^{ixS^0} \\
&+ \frac{1}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} \left[\left[\mathbf{U}(s) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{k+n,j+n} - \mathbb{E} \left[\mathbf{U}(s) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{k+n,j+n} \right] e^{ixS^0} \\
&+ \frac{ix}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} [\mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n,j+n} \frac{\partial S}{\partial Y_{jk}^{(2)}} e^{ixS^0} =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

For the first term \mathcal{J}_1 we may use (2.14) and get

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{n} \int_0^s \mathbb{E} [u_n(s_1) u'_n(s - s_1) - \mathbb{E} u_n(s_1) u'_n(s - s_1)] e^{ixS^0} ds_1 \\
&+ \frac{s}{n} \mathbb{E} [u'_n(s) - \mathbb{E} u'_n(s)] e^{ixS^0}.
\end{aligned}$$

Repeating the step (2.12) and (2.13) the last relation may be rewritten in the following way

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{n} \int_0^s [\mathbb{E} u_n(s_1) \mathbb{E} (u'_n)^0(s - s_1) e^{ixS^0} + \mathbb{E} u'_n(s - s_1) \mathbb{E} u_n^0(x, s_1) e^{ixS^0}] ds_1 \\
&+ \frac{s}{n} \mathbb{E} [u'_n(s) - \mathbb{E} u'_n(s)] e^{ixS^0}.
\end{aligned}$$

For the second term \mathcal{J}_2 we have

$$\mathcal{J}_2 = Y_n(x, s).$$

Let us consider now the term \mathcal{J}_3 . We have

$$\begin{aligned}
\mathcal{J}_3 &= \frac{ix}{\sqrt{n}} \sum_{j,k=1}^n \mathbb{E} [\mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n,j+n} \frac{\partial S}{\partial Y_{jk}^{(2)}} e^{ixS^0} \\
&= \frac{ix}{n} \sum_{j,k=1}^n \mathbb{E} [\mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n,j+n} [\mathbf{H}^{(2)} \mathbf{J} f'(\hat{\mathbf{V}})]_{j+n,k+n} e^{ixS^0} \\
&= \frac{x}{2n} \mathbb{E} \text{Tr} \mathbf{U}'(s) f'(\hat{\mathbf{V}}) e^{ixS^0}.
\end{aligned}$$

where we have applied the unitary matrix block decomposition (B.4) and used the following fact

$$\int_{-\infty}^{\infty} u \hat{f}(u) \sum_{k=1}^n [\mathbf{U}_3(s) \mathbf{W} \mathbf{H}(\Lambda(u) + \Lambda(-u)) \mathbf{H}^*]_{kk} = 0$$

which is valid for $f(\lambda)$ is an even function.

Finally we will have

$$\begin{aligned}
I_{11} = & -\frac{1}{n^2} \int_0^t ds \int_0^s \mathbb{E} u_n(s-s_1) \int_0^{s_1} [\mathbb{E} u_n(s_2) \mathbb{E} (u'_n)^0(s_1-s_2) e^{ixS^0} \\
& + \mathbb{E} u'_n(s_1-s_2) \mathbb{E} u_n^0(x, s_2) e^{ixS^0}] ds_2 ds_1 \\
& - \frac{1}{n^2} \int_0^t ds \int_0^s s_1 \mathbb{E} u_n(s-s_1) \mathbb{E} (u'_n(s_1))^0 ds_1 \\
& - \frac{1}{n} \int_0^t ds \int_0^s \mathbb{E} u_n(s_1) Y_n(x, s-s_1) ds_1 \\
& - \frac{x}{2n^2} \int_0^t ds \int_0^s \mathbb{E} u_n(s_1) \mathbb{E} \text{Tr} \mathbf{U}'(s-s_1) f'(\hat{\mathbf{V}}) e^{ixS^0} ds_1.
\end{aligned}$$

Changing the limits of integration, applying Lemma 2.5 and $\mathbb{E} |u_n(t)| \leq n$, we get

$$\begin{aligned}
I_{11} = & -\frac{2}{n^2} \int_0^t Y_n(x, s) ds \int_0^{t-s} \mathbb{E} u_n(s_1) \mathbb{E} u_n(t-s-s_1) ds_1 \\
& - \frac{x}{2n^2} \int_0^t \mathbb{E} u_n(s) \mathbb{E} \text{Tr}(\mathbf{U}(t-s) - \mathbf{I}) f'(\hat{\mathbf{V}}) e^{ixS^0} ds + r_n(t).
\end{aligned}$$

Finally for the term I_1 we may write the following representation.

$$\begin{aligned}
I_1 = & -\frac{3}{n^2} \int_0^t Y_n(x, s) ds \int_0^{t-s} \mathbb{E} u_n(s_1) \mathbb{E} u_n(t-s-s_1) ds_1 \\
& - \frac{x Z_n(x)}{2n^2} \int_0^t \mathbb{E} u_n(s) \mathbb{E} \text{Tr}(\mathbf{U}(t-s) - \mathbf{I}) f'(\hat{\mathbf{V}}) ds + r_n(t).
\end{aligned}$$

It remains to calculate the term I_2 .

$$\begin{aligned}
I_2 = & -\frac{x}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} \frac{\partial S}{\partial Y_{jk}^{(1)}} e^{ixS^0} ds \\
= & -\frac{x}{n} \int_0^t \sum_{j,k=1}^n \mathbb{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{j,k+n} e^{ixS^0} ds \\
= & -\frac{x Z_n(x)}{n} \int_0^t \sum_{k=1}^n \mathbb{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k,k+n} ds
\end{aligned}$$

where we have used the following observation. First of all we may write

$$\begin{aligned}
[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s) \mathbf{U}(u) \mathbf{H}^{(1)}]_{k,k+n} &= \sum_{j=1}^{2n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(u) \mathbf{H}^{(1)}]_{j,k+n} \\
&= \text{Tr}[(\mathbf{Y}^{(2)})^T \mathbf{Y}^{(2)} \mathbf{U}_3(s) \mathbf{U}_2(u)] + \text{Tr}[(\mathbf{Y}^{(2)})^T \mathbf{Y}^{(2)} \mathbf{U}_4(s) \mathbf{U}_4(u)] \\
&= \sum_{j=1}^n \mathbb{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(u) \mathbf{H}^{(1)}]_{j,k+n} + \text{Tr}[(\mathbf{Y}^{(2)})^T \mathbf{Y}^{(2)} \mathbf{U}_4(s) \mathbf{U}_4(u)].
\end{aligned}$$

From the representation (B.4) it follows that

$$\mathbf{U}_4(s) \mathbf{U}_4(u) = 4 \mathbf{H} \mathbf{D}(s, u) \mathbf{H}^*,$$

where $\mathbf{D}(s, u)$ is a diagonal matrix with $D_{jj}(s, u) = \cos(s_j s) \cos(s_j u)$, $j = 1, \dots, n$. Since $\hat{f}(t)$ is an even function we have

$$\int_{-\infty}^{\infty} u \hat{f}(u) \int_0^t \mathbb{E} \operatorname{Tr}[(\mathbf{Y}^{(2)})^T \mathbf{Y}^{(2)} \mathbf{H} \mathbf{D}(s, u) \mathbf{H}^*] ds du = 0.$$

We investigate now the behavior of

$$(2.15) \quad \sum_{k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k, k+n}.$$

Applying the same arguments as before

$$\begin{aligned} \mathbb{E} \sum_{k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k, k+n} &= \frac{1}{\sqrt{n}} \sum_{j, k=1}^n \mathbb{E} Y_{jk}^{(2)} [\mathbf{U}(t) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k+n, j+n} \\ &= \frac{1}{\sqrt{n}} \sum_{j, k=1}^n \mathbb{E} \left[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(2)}} f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)} \right]_{k+n, j+n} + \frac{1}{\sqrt{n}} \sum_{j, k=1}^n \mathbb{E} \left[\mathbf{U}(t) f'(\hat{\mathbf{V}}) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{k+n, j+n} \\ &=: T_1 + T_2. \end{aligned}$$

The term T_1 may be expanded in the sum of two terms

$$\begin{aligned} T_1 &= \frac{1}{\sqrt{n}} \sum_{j, k=1}^n \sum_{l=1}^{2n} \mathbb{E} \left[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(2)}} \right]_{k+n, l} [f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{l, j+n} \\ &= \frac{i}{n} \int_0^t \sum_{k=1}^n \mathbb{E} U_{k+n, k+n}(s) \sum_{j=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t-s) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{j+n, j+n} ds \\ &\quad + \frac{i}{n} \int_0^t \sum_{k, j=1}^n \mathbb{E} [\mathbf{U}(s) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k+n, j+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t-s)]_{j+n, k+n} \\ &= \frac{1}{2n} \int_0^t \mathbb{E} u_n(s) \operatorname{Tr} \mathbf{U}'(t-s) f'(\hat{\mathbf{V}}) ds \\ &\quad + \frac{i}{n} \int_0^t \sum_{k, j=1}^n \mathbb{E} [\mathbf{U}(s) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k+n, j+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t-s)]_{j+n, k+n}. \end{aligned}$$

For the term T_2 we get

$$T_2 = \sum_{k=1}^n \mathbb{E} [\mathbf{U}(t) f'(\hat{\mathbf{V}})]_{k+n, k+n} = \frac{1}{2} \mathbb{E} \operatorname{Tr} \mathbf{U}(t) f'(\hat{\mathbf{V}}).$$

We get the following decomposition for (2.15)

$$\begin{aligned} \mathbb{E} \sum_{k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k,k+n} &= \frac{1}{n} \int_0^t \mathbb{E} u_n(s) \operatorname{Tr} \mathbf{U}'(t-s) f'(\hat{\mathbf{V}}) ds \\ &+ \frac{1}{2} \operatorname{Tr} \mathbf{U}(t) f'(\hat{\mathbf{V}}) \\ &+ \frac{i}{n} \int_0^t \sum_{k,j=1}^n \mathbb{E} [\mathbf{U}(s) f'(\hat{\mathbf{V}}) \mathbf{H}^{(1)}]_{k+n,j+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t-s)]_{j+n,k+n}. \end{aligned}$$

Inserting this equation to I_2 we will have

$$\begin{aligned} I_2 &= -\frac{xZ_n(x)}{2n^2} \int_0^t \int_0^s \mathbb{E} u_n(s_1) \operatorname{Tr} \mathbf{U}'(s-s_1) f'(\hat{\mathbf{V}}) ds_1 ds \\ &\quad - \frac{xZ_n(x)}{2n} \int_0^t \operatorname{Tr} \mathbf{U}(s) f'(\hat{\mathbf{V}}) ds + r_n(t). \end{aligned}$$

Changing the limits of integration and applying Lemma 2.5, we get

$$\begin{aligned} I_2 &= -\frac{xZ_n(x)}{2n^2} \int_0^t \mathbb{E} u_n(s) \mathbb{E} \operatorname{Tr} [\mathbf{U}(t-s) f'(\hat{\mathbf{V}})] ds \\ &\quad - \frac{xZ_n(x)}{2n} \int_0^t \operatorname{Tr} \mathbf{U}(s) f'(\hat{\mathbf{V}}) ds + r_n(t), \end{aligned}$$

where we have also used that $f(\lambda)$ and $\mathbb{E} u(s)$ are even functions. It follows from (2.11) that we have derived representation for \mathcal{A}_1 . The same arguments are valid for \mathcal{A}_2 .

To simplify our notations let us introduce the following quantity

$$A_n(t) := -\frac{1}{2n} \mathbb{E} \operatorname{Tr} [\mathbf{U}(t) f'(\hat{\mathbf{V}})].$$

One may see that $A_n(t)$ depends on t only, but $Z_n(x)$ depends on x only. We derive an equation for $Y_n(x, t)$:

$$\begin{aligned} (2.16) \quad Y_n(x, t) &+ 3 \int_0^t Y_n(x, s) v_n^{2*}(t-s) ds \\ &= xZ_n(x) \int_0^t [v_n(s) A_n(t-s) + A_n(s)] ds + r_n(x, t), \end{aligned}$$

where

$$v_n(t) := \frac{1}{n} \mathbb{E} u_n(t).$$

As n goes to infinity the sequence $v_n(t)$ uniformly converges to the following function

$$v(t) = \int_{-a}^a e^{itx} p(x) dx,$$

where

$$p(x) := |x| P_2(x^2) \text{ and } a := \sqrt{K_2},$$

with $P_2(x)$, K_2 defined in Appendix A. This function is a characteristic function of Fuss-Catalan distribution. The same arguments lead to

$$A(t) := \lim_{n \rightarrow \infty} A_n(t) = - \int_{-a}^a e^{it\lambda} f'(\lambda) p(\lambda) d\lambda.$$

Taking a limit in (2.16) with respect to $n_l \rightarrow \infty$ we get

$$\begin{aligned} Y(x, t) + 3 \int_0^t Y(x, s) v^{2*}(t-s) ds \\ (2.17) \quad \quad \quad = xZ(x) \int_0^t [2v(s)A(t-s) + A(s)] ds, \end{aligned}$$

Denote by $F(z)$, $V(z)$ and $R(z)$ the generalized Fourier transform of $Y(x, t)$, $v(t)$ and $A(t)$ respectively (see Appendix D. Applying Statement D.1 we get from (2.17)

$$F(z) - 3F(z)V^2(z) = -2ixZ(x)R(z)V(z) + \frac{ixZ(x)R(z)}{z}.$$

and it follows that

$$(2.18) \quad F(z) = \frac{ixZ(x)(-2R(z)V(z) + R(z)/z)}{1 - 3V^2(z)}.$$

It is easy to check that

$$V(z) = s(z),$$

where $s(z)$ is the Stieltjes transform of $p(x)$. In these notations we may rewrite (2.18) as follows

$$(2.19) \quad F(z) = \frac{ixZ(x)(-2R(z)s(z) + R(z)/z)}{1 - 3s^2(z)}.$$

By Lemma D.2 the inverse Fourier transform of

$$\frac{1/z - 2s(z)}{1 - 3s^2(z)}$$

is given by

$$(2.20) \quad T(t) = \frac{1}{\pi} \int_{-a}^a \frac{e^{it\mu}}{3p_1(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3} d\mu,$$

where $p_1(\mu) := \pi p(\lambda)$. From (2.19) and (2.20) we conclude

$$\begin{aligned} Y(x, t) = -\frac{xZ(x)}{\pi^2} \int_0^t \int_{-a}^a e^{is\lambda} f'(\lambda) p(\lambda) d\lambda \\ \times \int_{-a}^a \frac{e^{i(t-s)\mu}}{3p(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3} dv \end{aligned}$$

Simple calculation yields

$$\begin{aligned} Y(x, t) = \frac{ixZ(x)}{\pi^2} \int_{-a}^a p(\lambda) d\lambda \\ (2.21) \quad \quad \times \int_{-a}^a \frac{e^{it\lambda} - e^{it\mu}}{\lambda - \mu} f'(\lambda) \frac{1}{3p(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3} d\mu. \end{aligned}$$

Finally we get from (2.10) and (2.21)

$$(2.22) \quad \begin{aligned} \lim_{n_l \rightarrow \infty} Z'_n(x, t) &= -\frac{xZ(x)}{\pi^2} \int_{-a}^a p(\lambda) d\lambda \\ &\times \int_{-a}^a \frac{f(\lambda) - f(\mu)}{\lambda - \mu} f'(\lambda) \frac{1}{3p(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3} d\mu. \end{aligned}$$

One may see that

$$(f(\lambda) - f(\mu))f'(\lambda) = \frac{1}{2} \frac{\partial}{\partial \lambda} (f(\lambda) - f(\mu))^2,$$

and (2.22) may be rewritten applying integration by parts in the following way

$$\begin{aligned} \lim_{n_l \rightarrow \infty} Z'_n &= -\frac{xZ(x)}{2\pi^2} \int_{-a}^a \int_{-a}^a \frac{(f(\lambda) - f(\mu))^2}{(\lambda - \mu)^2} \\ &\times \frac{[p(\lambda) - p'(\lambda)(\lambda - \mu)]}{3p(\mu)} \frac{[4p_1(\mu)^4 + 11p_1(\mu)^2 + 4]}{4p_1(\mu)^2 + 3} d\lambda d\mu. \end{aligned}$$

Comparing this with (2.9) we may conclude the proof of Theorem 2.1. \square

3. THE GENERAL CASE

In this section we finish the proof of Theorem 1.1. Applying the method of Bentkus from [8] and the method of Tikhomirov from [25], [26] we show that one may substitute the general matrix by the matrix with i.i.d. Gaussian random variables and express the characteristic function in the general case via the characteristic function in the Gaussian case. These methods have been applied several times in random matrix theory, see, for example, [17] and [16].

3.1. Truncation. In this subsection we show by standard arguments that we may truncate the entries of $\mathbf{X}^{(q)}$, $q = 1, 2$. For all $1 \leq j, k \leq n$, $q = 1, 2$, we introduce truncated random variables $\hat{X}_{jk}^{(q)} := X_{jk}^{(q)} \mathbb{I}(|X_{jk}^{(q)}| \geq \tau\sqrt{n})$. Denote by $\hat{\mathbf{X}}^{(q)} := [\hat{X}_{jk}^{(q)}]_{j,k=1}^n$. One may see that

$$(3.1) \quad \mathbb{P}(\hat{\mathbf{X}}^{(q)} \neq \mathbf{X}^{(q)}) \leq \sum_{j,k=1}^n \mathbb{E} \mathbb{I}(|X_{jk}^{(q)}| \geq \tau\sqrt{n}) \leq \frac{1}{\tau^4} \mathbb{E} X_{11}^4 \mathbb{I}(|X_{11}^{(q)}| \geq \tau\sqrt{n}).$$

Let \hat{S}^0 denote S^0 with all $X_{jk}^{(q)}$ replaced by $\hat{X}_{jk}^{(q)}$. It follows from (3.1) that

$$\lim_{n \rightarrow \infty} |\mathbb{E} e^{itS^0} - \mathbb{E} e^{it\hat{S}^0}| = 0.$$

By the similar arguments one may show that we may assume that $\mathbb{E} \hat{X}_{jk}^{(q)} = 0$ and $\text{Var}(\hat{X}_{jk}^{(q)}) = 1$. In what follows we will assume that $|X_{jk}^{(q)}| \leq \tau\sqrt{n}$.

Remark. It is easy to see that one may assume $X_{jk}^{(q)}$ non i.i.d., but imply the following Lendberg type condition on the fourth moments

$$\text{for all } \tau > 0 \quad \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{j,k=1}^n \mathbb{E}(X_{jk}^{(q)})^4 \mathbb{I}(|X_{jk}^{(q)}| \geq \tau \sqrt{n}) = 0.$$

3.2. From the general case to the Gaussian case. Let $\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}$ be $n \times n$ independent random matrices with independent Gaussian entries $n^{-1/2}Y_{jk}^{(q)}$ such that

$$\mathbb{E}Y_{jk}^{(q)} = 0, \quad \mathbb{E}(Y_{jk}^{(q)})^2 = 1, \quad \text{for any } q = 1, 2, j, k = 1 \dots, n.$$

For any $\varphi \in [0, \frac{\pi}{2}]$ and any $\nu = 1, 2$, introduce the matrices

$$\mathbf{Z}^{(q)}(\varphi) = \mathbf{X}^{(q)} \sin \varphi + \mathbf{Y}^{(q)} \cos \varphi$$

where

$$[\mathbf{Z}^{(q)}(\varphi)]_{jk} = \frac{1}{\sqrt{n}} Z_{jk}^{(q)} = \frac{1}{\sqrt{n}} (X_{jk}^{(q)} \sin \varphi + Y_{jk}^{(q)} \cos \varphi).$$

We define the matrices $\mathbf{W}(\varphi), \mathbf{H}^{(q)}(\varphi), \mathbf{V}(\varphi), \hat{\mathbf{V}}(\varphi), \mathbf{U}(\varphi, t)$ by

$$\begin{aligned} \mathbf{W}(\varphi) &= \prod_{q=1}^2 \mathbf{Z}^{(q)}(\varphi), \quad \mathbf{H}^{(q)}(\varphi) = \begin{bmatrix} \mathbf{Z}^{(q)}(\varphi) & \mathbf{O} \\ \mathbf{O} & \mathbf{Z}^{(m-q+1)}(\varphi) \end{bmatrix} \\ \mathbf{V}(\varphi) &= \prod_{q=1}^2 \mathbf{H}^{(q)}(\varphi), \quad \hat{\mathbf{V}}(\varphi) = \mathbf{V}(\varphi) \mathbf{J}, \quad \mathbf{U}(\varphi, t) = e^{-it\hat{\mathbf{V}}} \end{aligned}$$

Recall that \mathbf{I} (with sub-index or without it) denotes the unit matrix of corresponding order, $\mathbf{J} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{bmatrix}$ and \mathbf{O} denotes the matrix with zero-entries.

Let $S(\varphi) = \text{Tr } f(\mathbf{Z}(\varphi)), S^0(\varphi) = S(\varphi) - \mathbb{E} S(\varphi)$ and

$$g_n(t, \varphi) = \mathbb{E} e^{itS^0(\varphi)}$$

Theorem 3.1. *Under (C0) $\lim_{n \rightarrow \infty} g_n(t)$ satisfies the following equation*

$$g(t, \pi/2) = g(t, 0) e^{-t^2 \kappa_4 \Psi^2/4},$$

where

$$(3.2) \quad \Psi = \int_{-a}^a f(\lambda) [p(\lambda) + \lambda p'(\lambda)] d\lambda$$

and $\kappa_4 = \mu_4 - 3$.

Proof. We prove that the function $\lim_{n \rightarrow \infty} g_n(t, \varphi)$ satisfies the following equation

$$\frac{\partial g(t, \varphi)}{\partial \varphi} = -\kappa_4 t^2 \sin^3 \varphi \cos \varphi \Psi^2 g(t, \varphi).$$

It will follow from this equation that

$$g(t, \pi/2) = g(t, 0) \exp \left\{ -\kappa_4 t^2 \Psi^2 \int_0^{\pi/2} \sin^3 \alpha \cos \alpha d\alpha \right\}$$

Note that

$$g_n(t, \pi/2) - g_n(t, 0) = \int_0^{\pi/2} \frac{\partial g_n(t, \varphi)}{\partial \varphi} d\varphi$$

Similarly to the section 2 it is sufficient to prove that any converging subsequences $\{g_{n_l}\}$ and $\left\{\frac{\partial g_{n_l}}{\partial \varphi}\right\}$ satisfy

$$\lim_{n_l \rightarrow \infty} g_{n_l}(t, \varphi) = g(t, \varphi), \quad \lim_{n_l \rightarrow \infty} \frac{\partial g_{n_l}(t, \varphi)}{\partial \varphi} = -\kappa_4 t^2 \sin^3 \varphi \cos \varphi \Psi^2 g(t, \varphi).$$

By Lemma B.5 we get

$$(3.3) \quad \frac{\partial g_n(t, \varphi)}{\partial \varphi} = \frac{it}{\sqrt{n}} \sum_{q=1}^2 \sum_{j,k=1}^n \mathbb{E} \hat{Z}_{jk}^{(q)} [\mathbf{V}_{[m-q+2, m]} \mathbf{J} f'(\hat{\mathbf{V}}) \mathbf{V}_{[1, m-q]}]_{j+n, k+n} e^{itS^0},$$

where

$$\hat{Z}_{jk}^{(q)} := \frac{d}{d\varphi} Z_{jk}^{(q)} = X_{jk}^{(q)} \cos \varphi - Y_{jk}^{(q)} \sin \varphi.$$

It is straightforward to check that

$$(3.4) \quad \mathbb{E} \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^p = 0, \text{ for } p = 0, 1;$$

$$\mathbb{E} \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^2 = \mathbb{E} (X_{jk}^{(q)})^3 \cos^3 \varphi;$$

$$(3.5) \quad \mathbb{E} \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^3 = \kappa_4 \sin^3 \varphi \cos \varphi.$$

Let us introduce further notations. Denote by $\mathbf{V}_{[\alpha, \beta]}^{j, k, q}(x)$ the corresponding matrix $\mathbf{V}_{[\alpha, \beta]}$ with $Z_{jk}^{(q)}$ replaced by x . Let us also denote

$$\Phi_{jkq}(x) := [\mathbf{V}_{[m-q+2, m]}^{(j, k, q)}(x) \mathbf{J} f'(\hat{\mathbf{V}}^{(j, k, q)}(y)) \mathbf{V}_{[1, m-q]}^{(j, k, q)}(x)]_{j+n, k+n} e^{itS^0(\mathbf{V}^{(j, k, q)}(x))}.$$

Applying Taylor's formula we get

$$\Phi_{jkq}(Z_{jk}^{(q)}) = \sum_{p=0}^3 \frac{1}{p!} ((Z_{jk}^{(q)})^p \Phi_{jkq}^{(p)}(0) + \frac{1}{3!} (Z_{jk}^{(q)})^4 \mathbb{E}_\theta (1 - \theta)^3 \Phi_{jkq}^{(4)}(\theta Z_{jk}^{(q)}))$$

This equation and (3.3) together imply

$$\begin{aligned} \frac{\partial g_n(t, \varphi)}{\partial \varphi} &= \frac{it}{n^{1/2}} \sum_{p=1}^3 \frac{1}{p!} \sum_{q=1}^2 \sum_{j,k=1}^n \mathbb{E} \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^p \mathbb{E} \Phi_{jkq}^{(p)}(0) \\ &\quad + \frac{it}{3! n^{1/2}} \sum_{q=1}^2 \sum_{j,k=1}^n \mathbb{E} (1 - \theta)^3 \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^4 \Phi_{jkq}^{(4)}(\theta Z_{jk}^{(q)}) \\ &= T_1 + \dots + T_4. \end{aligned}$$

It follows from (3.4) that $T_1 = 0$. In the next subsections we will investigate the term $T_k, k = 2, 3, 4$.

3.3. The second derivative. First we note that

$$[\mathbf{V}_{[m-q+2,m]}^{(j,k,q)} \mathbf{J} \mathbf{A} \mathbf{V}_{[1,m-q]}^{(j,k,q)}]_{j+n,k+n} = [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \mathbf{A} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}.$$

for an arbitrary matrix \mathbf{A} . It is straightforward to check that

$$\Phi_{jkq}^{(2)}(0) = L_{jkq}^1 + L_{jkq}^2 + L_{jkq}^3,$$

where

$$\begin{aligned} L_{jkq}^1 &= [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^2 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^2} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0}, \\ L_{jkq}^2 &= \frac{3it}{\sqrt{n}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} \\ &\quad \times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0}, \\ L_{jkq}^3 &= -\frac{t^2}{n} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^3 e^{itS^0}. \end{aligned}$$

Let us consider, for example, the term $\frac{1}{\sqrt{n}} L_{jkq}^2$. We have, for $q = 1$,

$$\frac{1}{n^{1/2}} \sum_{j,k=1}^n \mathbb{E} L_{jkq}^2 = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \frac{3t}{n^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uv \hat{f}(u) \hat{f}(v) \int_0^u \mathbb{E} \sum_{j,k=1}^n [\mathbf{U}(s) \mathbf{H}^{(1)}]_{j,k+n} \\ &\quad \times [\mathbf{U}(u-s) \mathbf{H}^{(1)}]_{j,k+n} [\mathbf{U}(v) \mathbf{H}^{(1)}]_{j,k+n} ds du dv \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{3t}{n^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uv \hat{f}(u) \hat{f}(v) \int_0^u \mathbb{E} \sum_{j,k=1}^n [(\mathbf{H}^{(1)})^T \mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n,k+n} \\ &\quad \times [\mathbf{U}(u-s)]_{j,j} [\mathbf{U}(v) \mathbf{H}^{(1)}]_{j,k+n} ds du dv \end{aligned}$$

We estimate the term I_1 . Applying Cauchy-Schwarz inequality and orthogonality relations for \mathbf{U} we get

$$\begin{aligned}
& \frac{1}{n^{3/2}} \mathbb{E} \left| \sum_{j,k=1}^n [\mathbf{U}(s)\mathbf{H}^{(1)}]_{j,k+n} [\mathbf{U}(u-s)\mathbf{H}^{(1)}]_{j,k+n} [\mathbf{U}(v)\mathbf{H}^{(1)}]_{j,k+n} \right| \\
& \leq \frac{1}{n^3} \mathbb{E} \left[\sum_{l_1, l_2, l_3=1}^n \left(\sum_{k=1}^n Z_{kl_1}^{(2)} Z_{kl_2}^{(2)} Z_{kl_3}^{(2)} \right) \left(\sum_{j=1}^n [\mathbf{U}_2]_{jl_1}(s) [\mathbf{U}_2]_{jl_2}(u-s) [\mathbf{U}_2]_{jl_3}(v) \right) \right] \\
& \leq \frac{1}{n^{5/2}} \mathbb{E}^{1/2} \left[\sum_{l_1, l_2, l_3=1}^n \left(\sum_{k=1}^n Z_{kl_1}^{(2)} Z_{kl_2}^{(2)} Z_{kl_3}^{(2)} \right)^2 \right] \\
& \leq \frac{1}{n^{5/2}} \mathbb{E}^{1/2} \left[\sum_{l=1}^n \left(\sum_{k=1}^n (Z_{kl}^{(2)})^3 \right)^2 + 3 \sum_{l_1 \neq l_2} \left(\sum_{k=1}^n (Z_{kl_1}^{(2)})^2 Z_{kl_2}^{(2)} \right)^2 \right. \\
& \quad \left. + \sum_{l_1 \neq l_2 \neq l_3} \sum_{k=1}^n (Z_{kl_1}^{(2)})^2 (Z_{kl_2}^{(2)})^2 (Z_{kl_3}^{(2)})^2 \right] \\
& \leq C(\tau + 1)n^{-1/2}.
\end{aligned}$$

For the term I_2 we may apply the same arguments. Finally

$$|I_1 + I_2| \leq \frac{Ct(\tau + 1)}{n^{1/2}} \int_{-\infty}^{\infty} |u|^3 |\hat{f}(u)| du \int_{-\infty}^{\infty} |v| |\hat{f}(v)| dv$$

Analogously one may show that $\frac{1}{n^{1/2}} \sum_{j,k=1}^n L_{jkq}^1$ and $\frac{1}{n^{1/2}} \sum_{j,k=1}^n L_{jkq}^3$ goes to zero as n goes to infinity. It follows that $T_2 = o(1)$.

3.4. The third derivative. We investigate now the term T_3 . Direct computations yield

$$\begin{aligned}
\Phi_{jkq}^{(3)}(0) &= [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^3 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^3} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0} \\
&+ \frac{4it}{\sqrt{n}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^2 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^2} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} \\
&\times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}})] \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0} \\
&+ \frac{3it}{\sqrt{n}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^2 e^{itS^0} \\
&- \frac{6t^2}{n} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}})] \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^2 \\
&\times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0} \\
&- \frac{it^3}{n^{3/2}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}})] \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^4 e^{itS^0}
\end{aligned}$$

It is straightforward to check that all terms except the third are of order $o(1)$. These may be done similarly to the previous section. Let us denote

$$\Psi_n^q = \frac{1}{n} \sum_{j,k=1}^n \mathbb{E} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=0} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^2 e^{itS^0}.$$

Our aim is to show that

$$(3.6) \quad \lim_{n_l \rightarrow \infty} \Psi_{n_l}^q = \left[\int_{-a}^a f(\lambda) [p(\lambda) + \lambda p'(\lambda)] d\lambda \right]^2.$$

Consider the case $q = 1$. We get by Lemma B.3

$$\begin{aligned}
&\left[\frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(1)}} \mathbf{H}^{(1)} \right]_{j,k+n} = \sum_{l=1}^{2n} \left[\frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(1)}} \right]_{j,l} [\mathbf{H}^{(1)}]_{l,k+n} \\
&= -\frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} s \hat{f}(s) \sum_{l=1}^{2n} [\mathbf{U} \mathbf{H}^{(1)}]_{j,k+n} * [\mathbf{U}]_{l,j}(s) [\mathbf{H}^{(1)}]_{l,k+n} \\
&\quad + \sum_{l=1}^{2n} [\mathbf{U} \mathbf{H}^{(1)}]_{l,k+n} * [\mathbf{U}]_{j,j}(s) [\mathbf{H}^{(1)}]_{l,k+n} ds \\
&= -\frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} s \hat{f}(s) [\mathbf{U} \mathbf{H}^{(1)}]_{j,k+n} * [\mathbf{U} \mathbf{H}^{(1)}]_{j,k+n}(s) \\
&\quad + [(\mathbf{H}^{(1)})^T \mathbf{U} \mathbf{H}^{(1)}]_{k+n,k+n} * [\mathbf{U}]_{j,j}(s) ds
\end{aligned}$$

Similarly to the previous sections we may show that the first term in the last equation has the zero impact. It is straightforward to check

$$[(\mathbf{H}^{(1)})^T \mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n, k+n} = [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s) \mathbf{H}^{(1)}]_{k, k+n} =: T_k(s).$$

Let us investigate the following integral

$$\int_{-\infty}^{\infty} s \hat{f}(s) \mathbb{E} T_k * \mathbb{E}[\mathbf{U}]_{j,j}(s) ds$$

We have

$$\begin{aligned} \mathbb{E} T_k(s) &= \frac{1}{\sqrt{n}} \sum_{l=1}^n \mathbb{E} Z_{kl}^{(2)} [\mathbf{U}(s) \mathbf{H}^{(1)}]_{l+n, k+n} \\ &= \frac{1}{\sqrt{n}} \sum_{l=1}^n \mathbb{E} \left[\frac{\partial \mathbf{U}(s)}{\partial Z_{kl}^{(2)}} \mathbf{H}^{(1)} \right]_{l+n, k+n} + \frac{1}{\sqrt{n}} \sum_{l=1}^n \mathbb{E} \left[\mathbf{U}(s) \frac{\partial \mathbf{H}^{(1)}(s)}{\partial Z_{kl}^{(2)}} \right]_{l+n, k+n}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{l=1}^n \mathbb{E} \left[\frac{\partial \mathbf{U}(s)}{\partial Z_{kl}^{(2)}} \mathbf{H}^{(1)} \right]_{l+n, k+n} &= \sum_{l=1}^n \sum_{p=1}^{2n} \mathbb{E} \left[\frac{\partial \mathbf{U}(s)}{\partial Z_{kl}^{(2)}} \right]_{l+n, p} [\mathbf{H}^{(1)}]_{p, k+n} \\ &= \frac{i}{\sqrt{n}} \sum_{l=1}^n \sum_{p=1}^{2n} \mathbb{E}[\mathbf{U}]_{l+n, l+n} * [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}]_{k+n, p} [\mathbf{H}^{(1)}]_{p, k+n} \\ &+ \frac{i}{\sqrt{n}} \sum_{l=1}^n \sum_{p=1}^{2n} \mathbb{E}[\mathbf{U}]_{p, l+n} * [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}]_{k+n, l+n} [\mathbf{H}^{(1)}]_{p, k+n} \\ &= \frac{i}{\sqrt{n}} \sum_{l=1}^n \mathbb{E}[\mathbf{U}]_{l+n, l+n} * [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U} \mathbf{H}^{(1)}]_{k+n, k+n} \\ &+ \frac{i}{\sqrt{n}} \sum_{l=1}^n \mathbb{E}[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}]_{k+n, l+n} * [\mathbf{U} \mathbf{H}^{(1)}]_{l+n, k+n}. \end{aligned}$$

We may show that the second term has the zero impact. It follows that

$$(3.7) \quad \mathbb{E} T_k(s) = \frac{i}{n} \mathbb{E} u_n * \mathbb{E}[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U} \mathbf{H}^{(1)}]_{k+n, k+n}(s) + \frac{1}{n} \mathbb{E} u_n(s) + o(1),$$

where we have also applied Lemma C.2. Finally we will have for $q = 1$

$$(3.8) \quad \Psi_n^q = \frac{g_n(\varphi, t)}{n^2} \sum_{j,k=1}^n \left\{ \int_{-\infty}^{\infty} s \hat{f}(s) \left[\frac{1}{n} \mathbb{E} u_n * \mathbb{E}[\mathbf{U}]_{j,j}(s) + \frac{i}{n} \mathbb{E} u_n * \mathbb{E}[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U} \mathbf{H}^{(1)}]_{k+n, k+n} * \mathbb{E}[\mathbf{U}]_{j,j}(s) \right] ds \right\}^2 + o(1).$$

Let us introduce further notations and denote

$$V_{n,k}(s) := \mathbb{E}[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s) \mathbf{H}^{(1)}]_{k+n, k+n}.$$

We may write, applying Lemma B.3

$$\begin{aligned}
V_{n,k} &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \mathbb{E} Z_{jk}^{(1)} [\mathbf{U} \mathbf{H}^{(1)}]_{j,k+n} = \sum_{j=1}^n \mathbb{E} \left[\frac{\partial \mathbf{U}(s)}{\partial Z_{jk}^{(1)}} \right]_{j,k+n} \\
&= \frac{i}{n} \sum_{j=1}^n \sum_{l=1}^{2n} \mathbb{E} [\mathbf{U} \mathbf{H}^{(1)}]_{j,k+n} * [\mathbf{U}]_{l,j}(s) [\mathbf{H}^{(1)}]_{l,k+n} \\
&\quad + \frac{i}{n} \sum_{j=1}^n \sum_{l=1}^{2n} \mathbb{E} [\mathbf{U} \mathbf{H}^{(1)}]_{l,k+n} * [\mathbf{U}]_{j,j}(s) [\mathbf{H}^{(1)}]_{l,k+n}
\end{aligned}$$

The same arguments as before yield that

$$(3.9) \quad V_{n,k}(s) = \frac{i}{n} \mathbb{E} u_n(s) * \mathbb{E} T_k(s) + o(1).$$

Applying (3.7) we get

$$V_{n,k}(s) = \frac{i}{n^2} \mathbb{E} u_n * \mathbb{E} u_n(s) - \frac{1}{n^2} \mathbb{E} u_n * \mathbb{E} u_n * V_{n,k}(s) + o(1).$$

This means that $\lim_{n \rightarrow \infty} V_{n,k}$ satisfies the following equation

$$h(s) = iu * u(s) - u * u * h(s)$$

The same equation holds for $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n V_{n,k}(s)$. Since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n V_{n,k}(s) = -iv'(s)$$

that means that

$$\lim_{n \rightarrow \infty} V_{n,k} = -iv'(s)$$

Taking the limit with respect to $n_l \rightarrow \infty$ we get in (3.8)

$$(3.10) \quad \lim_{n_l \rightarrow \infty} \Psi_n^q = g(\varphi, t) \left\{ \int_{-\infty}^{\infty} s \hat{f}(s) [v * v(s) + v * v * v'(s)] ds \right\}^2$$

Let us consider the following integral

$$\int_{-\infty}^{\infty} s \hat{f}(s) [v * v(s) + v * v * v'(s)] ds$$

First we investigate the Fourier transform of $v * v(s) + v * v * v'(s)$. It is given by

$$is^2(z) - i(1 + zs(z))s^2(z) = -izs^3(z)$$

By Proposition D.1 it follows that

$$(3.11) \quad v * v(t) + v * v * v'(s) = \frac{1}{2\pi} \int_L e^{isz} zs^3(z) dz.$$

Since $1 + zs(z) = zs^3(z)$, the right hand side of (3.11) may be rewritten as

$$\frac{1}{2\pi} \int_L e^{isz} (1 + zs(z)) dz.$$

Similarly to the proof of Lemma D.2 we get

$$\frac{1}{2\pi} \int_L e^{isz}(1 + zs(z)) dz = i \int_{-a}^a e^{is\lambda} \lambda p(\lambda) d\lambda.$$

Integrating by parts we will have

$$i \int_{-a}^a e^{is\lambda} \lambda p(\lambda) d\lambda = -\frac{1}{s} \int_{-a}^a e^{is\lambda} [p(\lambda) + \lambda p'(\lambda)] d\lambda.$$

Finally we conclude that

$$\int_{-\infty}^{\infty} t \hat{f}(t) [v * v(s) + v * v * v'(s)] ds = \int_{-a}^a f(\lambda) [p(\lambda) + \lambda p'(\lambda)] d\lambda.$$

and finish the proof of (3.6). If we show that for all $1 \leq j, k \leq n$

$$(3.12) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{-\infty}^{\infty} s \hat{f}(s) [T_k * [\mathbf{U}]_{jj}(s) - \mathbb{E} T_k * \mathbb{E} [\mathbf{U}]_{jj}(s)] ds \right]^2 e^{itS^0} = 0.$$

then from (3.5) and (3.6) we will have

$$\lim_{n_l \rightarrow \infty} \frac{it}{3!n_l^{1/2}} \sum_{q=1}^2 \sum_{j,k=1}^{n_l} \mathbb{E} \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^3 \mathbb{E} \Phi_{jkq}^{(p)}(0) = -\kappa_4 t^2 \sin^3 \varphi \cos \varphi \Psi^2 g(t, \varphi),$$

where Ψ is given by (3.2).

To prove (3.12) it is enough to show that

$$(3.13) \quad \text{Var}[T_k(s)] = o(1), \quad k = 1, \dots, n;$$

$$(3.14) \quad \text{Var}([\mathbf{U}]_{jj}) = o(1), \quad j = 1, \dots, n.$$

We may apply Lemma C.2 to conclude the proof of Theorem.

3.5. The remainder term. To conclude the proof of Theorem 1.1 it remains to estimate the remainder term T_4 . One may see that $\mathbb{E} \hat{Z}_{jk}^{(q)} (Z_{jk}^{(q)})^4 \leq C\tau \sqrt{n} \mu_4$. Let Z be a random variable which has the same distribution as $Z_{11}^{(1)}$. We estimate $\mathbb{E} \sup_Z \Phi_{jkq}^{(3)}(Z)$. Simple calculations yield that

$$\Phi_{jkq}^{(3)}(Z) = L_{jkq}^1 + \dots + L_{jkq}^7,$$

where

$$\begin{aligned}
L_{jkq}^1 &= \frac{1}{n^2} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^4 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^4} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0}, \\
L_{jkq}^2 &= \frac{5it}{\sqrt{n}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^3 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^3} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} \\
&\quad \times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0}, \\
L_{jkq}^3 &= \frac{10it}{\sqrt{n}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^2 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^2} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} \\
&\quad \times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0}, \\
L_{jkq}^4 &= -\frac{10t^2}{n} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial^2 f'(\hat{\mathbf{V}})}{\partial (Z_{jk}^{(q)})^2} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} \\
&\quad \times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^2 e^{itS^0}, \\
L_{jkq}^5 &= -\frac{15t^2}{n} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^2 \\
&\quad \times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} e^{itS^0}, \\
L_{jkq}^6 &= -\frac{10it^3}{n^{3/2}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \frac{\partial f'(\hat{\mathbf{V}})}{\partial Z_{jk}^{(q)}} \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n} \\
&\quad \times [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^3 e^{itS^0}, \\
L_{jkq}^7 &= \frac{t^4}{n^2} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \Big|_{Z_{jk}^{(q)}=Z} \mathbf{V}_{[1,m-q]}]_{j+n,k+n}^5 e^{itS^0}.
\end{aligned}$$

Applying the same arguments as before in subsections 3.3 and 3.4 we get that

$$|T_4| \leq C\tau.$$

One may show that it is possible to change τ by the sequence τ_n , such that $\lim_{n \rightarrow \infty} \tau_n = 0$ and $\lim_{n \rightarrow \infty} \sqrt{n}\tau_n = \infty$. This fact finishes the proof of Theorem 3.1. \square

APPENDIX A. FUSS-CATALAN DISTRIBUTION

For any $m \in \mathbb{N}$ let us consider the sequence of numbers

$$M_k = \frac{1}{mk+1} \binom{k}{mk+k}, \quad k \in \mathbb{N} \cup \{0\}.$$

These numbers are called Fuss-Catalan numbers. In [22] the density function $P_m(x)$ which satisfy

$$\int_0^{K_m} x^k P_m(x) dx = M_k$$

were found. Here $K_m := (m+1)^{m+1}/m^m$. The explicit formula for $P_m(x)$ is given by the following formula

$$P_m(x) = \sum_{k=1}^m \Lambda_{k,m} x^{\frac{k}{m+1}-1} \times \\ {}_mF_{m-1} \left(\left[\left\{ 1 - \frac{1+j}{m} + \frac{k}{m+1} \right\}_{j=1}^m \right], \right. \\ \left. \left[\left\{ 1 + \frac{k-j}{m+1} \right\}_{j=1}^{k-1}, \left\{ 1 + \frac{k-j}{m+1} \right\}_{j=k+1}^m \right]; \frac{m^m}{(m+1)^{m+1}} x \right).$$

where the coefficients $\Lambda_{k,m}$ read for $k = 1, 2, \dots, m$

$$\Lambda_{k,m} := m^{-3/2} \sqrt{\frac{m+1}{2\pi}} \left(\frac{m^{m/(m+1)}}{m+1} \right)^k \frac{\left[\prod_{j=1}^{k-1} \Gamma\left(\frac{j-k}{m+1}\right) \right] \left[\prod_{j=k+1}^m \Gamma\left(\frac{j-k}{m+1}\right) \right]}{\prod_{j=1}^m \Gamma\left(\frac{j+1}{m} - \frac{k}{m+1}\right)}.$$

For example,

$$P_1(x) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}}.$$

and

$$P_2(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{\left[\sqrt[3]{2} (27 + 3\sqrt{81-12x})^{\frac{2}{3}} - 6\sqrt[3]{x} \right]}{x^{\frac{2}{3}} (27 + 3\sqrt{81-12x})^{\frac{1}{3}}},$$

valid for $x \in [0, 27/4]$.

APPENDIX B. UNITARY MATRIX DERIVATIVES

In this section we collect usefull facts about matrix derivatives and matrix exponent. Let us consider a function $f(\lambda)$ and denote by

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) e^{-it\lambda} d\lambda$$

its Fourier transform. Function $f(\lambda)$ may be reconstucted from $\hat{f}(t)$ via inverse Fourier transform

$$f(\lambda) = \int_{-\infty}^{\infty} \hat{f}(t) e^{it\lambda} dt.$$

Let $f^{(k)}(\lambda)$ be k -th derivative of $f(\lambda)$. Then

$$(B.1) \quad f^{(k)}(\lambda) = i^k \int_{-\infty}^{\infty} t^k \hat{f}(t) e^{it\lambda} dt.$$

We introduce further notations. Let \mathbf{M} an arbitrary symmetric matrix and M_{jk} be its entries. We denote

$$D_{jk} := \partial/\partial M_{jk};$$

$$\mathbf{U}(t) := e^{it\mathbf{M}}, U_{jk}(t) = (\mathbf{U}(t))_{jk}.$$

Then we may write

$$(B.2) \quad f(\mathbf{M}) = \int_{-\infty}^{\infty} \hat{f}(t) \mathbf{U}(t) dt.$$

We will use the following formula

$$(B.3) \quad e^{(\mathbf{M}_1 + \mathbf{M}_2)t} = e^{\mathbf{M}_1 t} + \int_0^t e^{\mathbf{M}_1(t-s)} \mathbf{M}_2 e^{(\mathbf{M}_1 + \mathbf{M}_2)s} ds,$$

valid for arbitrary matrices $\mathbf{M}_1, \mathbf{M}_2$ and $t \in \mathbb{R}$.

In what follows we shall use matrix notation (2.3) and (2.4). Consider the singular value decomposition of the matrix \mathbf{Y} of dimension $n \times n$. Let \mathbf{L} and \mathbf{H} be unitary matrices of dimension $n \times n$. Let $\mathbf{\Lambda}$ be a diagonal matrix whose entries are the singular values of the matrix \mathbf{Y} . We have the following representation

$$\mathbf{Y} = \mathbf{L} \mathbf{\Lambda} \mathbf{H}^*$$

We introduce the following matrix

$$\mathbf{Z}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{L}^* & \mathbf{H}^* \\ \mathbf{L}^* & -\mathbf{H}^* \end{bmatrix}$$

It is straightforward to check that

$$\mathbf{Z}^* \mathbf{V} \mathbf{Z} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{O} \\ \mathbf{O} & -\mathbf{\Lambda} \end{bmatrix}$$

and

$$\mathbf{Z}^* \mathbf{U}(s) \mathbf{Z} = \mathbf{Z}^* \begin{bmatrix} \mathbf{\Lambda}(s) & \mathbf{O} \\ \mathbf{O} & \mathbf{\Lambda}(-s) \end{bmatrix} \mathbf{Z},$$

where $\mathbf{\Lambda}(s)$ is a diagonal matrix such that $[\mathbf{\Lambda}(s)]_{jj} = e^{[is\Lambda_{jj}]}$, $j = 1, \dots, n$. A simple calculation yields that

$$(B.4) \quad \mathbf{U}(s) = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_3 & \mathbf{U}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{L}(\mathbf{\Lambda}(s) + \mathbf{\Lambda}(-s))\mathbf{L}^* & \mathbf{L}(\mathbf{\Lambda}(s) - \mathbf{\Lambda}(-s))\mathbf{H}^* \\ \mathbf{H}(\mathbf{\Lambda}(s) - \mathbf{\Lambda}(-s))\mathbf{L}^* & \mathbf{H}(\mathbf{\Lambda}(s) + \mathbf{\Lambda}(-s))\mathbf{H}^* \end{bmatrix},$$

We also denote by $\mathbf{M}^{(j,k)}$ a matrix \mathbf{M} with M_{jk} removed. To calculate derivatives of $\mathbf{U}(s)$ we need the following lemma.

Lemma B.1. *Let $1 \leq j, k \leq n$ and $m \geq 2$. Then*

$$(B.5) \quad \left[\frac{\partial \hat{\mathbf{V}}}{\partial Y_{jk}^{(1)}} \right]_{a,b} = \frac{1}{\sqrt{n}} [\mathbf{V}_{[1,m-1]}]_{a,k+n} \mathbb{I}(b = j) + \frac{1}{\sqrt{n}} [\mathbf{V}_{[2,m]} \mathbf{J}]_{k,b} \mathbb{I}(a = j)$$

for any $1 \leq a, b \leq 2n$.

Proof. We decompose $\hat{\mathbf{V}}$ in the following way

$$\hat{\mathbf{V}} = \left((\mathbf{H}^{(jk)})^{(1)} + \frac{Y_{jk}^{(1)}}{\sqrt{n}} \mathbf{E}_{j,k} \right) \mathbf{V}_{[2,m-1]} \left((\mathbf{H}^{(jk)})^{(m)} + \frac{Y_{jk}^{(1)}}{\sqrt{n}} \mathbf{E}_{k+n,j+n} \right) \mathbf{J}.$$

It is easy to see

$$[\mathbf{V}_{[1,m-1]} \mathbf{E}_{k+n,j+n} \mathbf{J}]_{a,b} = [\mathbf{V}_{[1,m-1]}]_{a,k+n} \mathbb{I}(b = j),$$

$$[\mathbf{E}_{j,k} \mathbf{V}_{[2,m]} \mathbf{J}]_{a,b} = [\mathbf{V}_{[2,m]} \mathbf{J}]_{k,b} \mathbb{I}(a = j)$$

and

$$[\mathbf{E}_{j,k} \mathbf{V}_{[2,m-1]} \mathbf{E}_{k+n,j+n} \mathbf{J}]_{a,b} = [\mathbf{V}_{[2,m-1]}]_{k,k+n} \mathbb{I}(a = b = j) = 0.$$

□

We may generalize the last lemma on the case when the derivative is taken with respect to $Y_{jk}^{(q)}$, $q = 2, \dots, m$. We have the following lemma

Lemma B.2. *Let $1 \leq j, k \leq n$ and $m \geq 2$. Then*

$$(B.6) \quad \left[\frac{\partial \hat{\mathbf{V}}}{\partial Y_{jk}^{(q)}} \right]_{a,b} = \frac{1}{\sqrt{n}} [\mathbf{V}_{[1,m-q]}]_{a,k+n} [\mathbf{V}_{[m-q+2,m]} \mathbf{J}]_{j+n,b} + \frac{1}{\sqrt{n}} [\mathbf{V}_{[1,q-1]}]_{a,j} [\mathbf{V}_{[q+1,m]} \mathbf{J}]_{k,b}.$$

Proof. The proof is similar. □

Lemma B.3. *Let $1 \leq j, k \leq n$ and $m \geq 2$. Then*

$$\left[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} \right]_{x,y} = \frac{i}{\sqrt{n}} [\mathbf{U} \mathbf{V}_{[1,m-1]}]_{x,k+n} * [\mathbf{U}]_{y,j}(t) + \frac{i}{\sqrt{n}} [\mathbf{U} \mathbf{V}_{[1,m-1]}]_{y,k+n} * [\mathbf{U}]_{x,j}(t).$$

Proof. Using the chain rule we will have

$$\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} = \sum_{a=1}^n \sum_{b=n+1}^{2n} \frac{\partial \mathbf{U}(t)}{\partial \hat{V}_{ab}} \frac{\partial \hat{V}_{a,b}}{\partial Y_{jk}^{(1)}} + \sum_{a=n+1}^{2n} \sum_{b=1}^n \frac{\partial \mathbf{U}(t)}{\partial \hat{V}_{ab}} \frac{\partial \hat{V}_{a,b}}{\partial Y_{jk}^{(1)}}.$$

Applying Lemma B.1 we will have

$$\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} = \frac{1}{\sqrt{n}} \sum_{b=n+1}^{2n} \frac{\partial \mathbf{U}(t)}{\partial \hat{V}_{jb}} [\mathbf{V}_{[2,m]} \mathbf{J}]_{k,b} + \frac{1}{\sqrt{n}} \sum_{a=n+1}^{2n} \frac{\partial \mathbf{U}(t)}{\partial \hat{V}_{aj}} [\mathbf{V}_{[1,m-1]}]_{a,k+n}.$$

From (B.3) it follows that

$$\begin{aligned} \frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} &= \frac{i}{\sqrt{n}} \sum_{b=n+1}^{2n} U_{xj} * U_{by}(t) [\mathbf{V}_{[2,m]} \mathbf{J}]_{k,b} \\ &\quad + \frac{i}{\sqrt{n}} \sum_{a=n+1}^{2n} U_{x,a} * U_{jy}(t) [\mathbf{V}_{[1,m-1]}]_{a,k+n}. \end{aligned}$$

Since $[\mathbf{U} \mathbf{V}_{[1,m-1]}]_{y,k+n} = [\mathbf{V}_{[2,m]} \mathbf{J} \mathbf{U}]_{k,y}$ we get the statement of Lemma. \square

Lemma B.4. *Let $1 \leq j, k \leq n$ and $m \geq 2$. Then*

$$\begin{aligned} \left[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(q)}} \right]_{x,y} &= \frac{i}{\sqrt{n}} [\mathbf{U} \mathbf{V}_{1,[m-q]}]_{x,k+n} * [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \mathbf{U}]_{j+n,y}(t) \\ &\quad + \frac{i}{\sqrt{n}} [\mathbf{U} \mathbf{V}_{1,[m-q]}]_{y,k+n} * [\mathbf{V}_{[m-q+2,m]} \mathbf{J} \mathbf{U}]_{j+n,x}(t). \end{aligned}$$

Proof. The proof is similar. \square

The following lemma gives an expression for derivative of $S(\hat{\mathbf{V}}) := \frac{1}{2} \text{Tr } f(\hat{\mathbf{V}})$ with respect to $Y_{jk}^{(1)}$.

Lemma B.5. *Let $1 \leq j, k \leq n$ and $m \geq 2$. Then*

$$(B.7) \quad \frac{\partial S}{\partial Y_{jk}^{(1)}} = \frac{1}{\sqrt{n}} [f'(\hat{\mathbf{V}}) \mathbf{V}_{[1,m-1]}]_{j,k+n}.$$

Proof. It is easy to see that

$$\frac{\partial S}{\partial Y_{jk}^{(1)}} = \frac{1}{2} \int_{-\infty}^{\infty} \hat{f}(u) \text{Tr} \frac{\partial \mathbf{U}(u)}{\partial Y_{jk}^{(1)}} du.$$

Applying Lemma B.3 we get

$$\begin{aligned} \frac{\partial S}{\partial Y_{jk}^{(1)}} &= \frac{i}{2\sqrt{n}} \int_{-\infty}^{\infty} s \hat{f}(s) [\mathbf{U}(s) \mathbf{V}_{[1,m-1]}]_{j,k+n} ds \\ &\quad + \frac{i}{2\sqrt{n}} \int_{-\infty}^{\infty} s \hat{f}(s) [\mathbf{V}_{[2,m]} \mathbf{J} \mathbf{U}(s)]_{k,j} ds \end{aligned}$$

Applying the properties of \mathbf{V} and \mathbf{U} we get (B.7). \square

Lemma B.6. *Let $1 \leq j, k \leq n$ and $m \geq 2$. Then*

$$(B.8) \quad \frac{\partial S}{\partial Y_{jk}^{(q)}} = \frac{1}{\sqrt{n}} [\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\hat{\mathbf{V}}) \mathbf{V}_{[1,m-q]}]_{j+n,k+n}.$$

Proof. The proof is similar. \square

APPENDIX C. AUXILIARY LEMMAS

In this section we prove some auxiliary Lemmas.

The following Lemma gives an estimate for variance of

$$T_n(s, t) := \frac{1}{n} \sum_{j,k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n}.$$

Lemma C.1. *Under condition of Theorem 2.1 we have*

$$\text{Var}(T_n(t, s)) \leq \frac{C \max(t^2, (t-s)^2)}{n}.$$

Proof. Let us introduce the following matrices

$$\mathbf{H}^{(q,l)} = \mathbf{H}^{(q)} - \mathbf{e}_l \mathbf{e}_l^T \mathbf{H}^{(q)} - \mathbf{H}^{(q)} \mathbf{e}_l \mathbf{e}_l^T$$

$$\tilde{\mathbf{H}}^{(q,l)} = \mathbf{H}^{(q)} - \mathbf{e}_{l+n} \mathbf{e}_{l+n}^T \mathbf{H}^{(q)} - \mathbf{H}^{(q)} \mathbf{e}_{l+n} \mathbf{e}_{l+n}^T$$

where $q = 1, 2$ and $l = 1, \dots, n$. We define the following filtration

$$\mathcal{F}_{q,l} = \sigma\{Y_{i_1, i_2}^{(q)}, l < i_1, i_2 \leq n, Y_{i_3 i_4}^{(2)}, i_3, i_4 = 1, \dots, n\}.$$

We may rewrite the difference

$$\begin{aligned} & \mathbb{E} \sum_{j,k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n} - \sum_{k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n} \\ &= \sum_{q=1}^2 \sum_{l=1}^n (\mathbb{E}_{q,l} - \mathbb{E}_{q,l-1}), \end{aligned}$$

where $\mathbb{E}_{q,l}$ is the mathematical expectation with respect to $\mathcal{F}_{q,l}$. It is easy to see that $\mathcal{F}_{1,n} = \mathcal{F}_{2,0}$ and

$$\begin{aligned} & \mathbb{E}_{1,l} \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} \mathbf{U}^{(1,l)}(s)]_{k,j} [\mathbf{U}^{(1,l)}(t-s) \mathbf{H}^{(1,l)}]_{j,k+n} \\ &= \mathbb{E}_{1,l-1} \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} \mathbf{U}^{(1,l)}(s)]_{k,j} [\mathbf{U}^{(1,l)}(t-s) \mathbf{H}^{(1,l)}]_{j,k+n} \\ & \mathbb{E}_{2,l} \sum_{j,k=1}^n [\mathbf{H}^{(2,l)} \mathbf{J} \mathbf{U}^{(2,l)}(s)]_{k,j} [\mathbf{U}^{(2,l)}(t-s) \tilde{\mathbf{H}}^{(1,l)}]_{j,k+n} \\ &= \mathbb{E}_{2,l-1} \sum_{j,k=1}^n [\mathbf{H}^{(2,l)} \mathbf{J} \mathbf{U}^{(2,l)}(s)]_{k,j} [\mathbf{U}^{(2,l)}(t-s) \tilde{\mathbf{H}}^{(1,l)}]_{j,k+n} \end{aligned}$$

We consider the case $q = 1$ only. The case $q = 2$ is similar. We may write

$$\begin{aligned} & \sum_{j,k=1}^n [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n} \\ & - \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J}]_{k,j} [\mathbf{U}^{(1,l)}(t-s) \mathbf{H}^{(1,l)}]_{j,k+n} = \Theta_{1,l} + \Theta_{2,l} + \Theta_{3,l} + \Theta_{4,l}, \end{aligned}$$

where

$$\begin{aligned} \Theta_{1,l} &= \sum_{j,k=1}^n [(\mathbf{H}^{(2)} - \tilde{\mathbf{H}}^{(2,l)}) \mathbf{J} \mathbf{U}(s)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n}, \\ \Theta_{2,l} &= \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} (\mathbf{U}(s) - \mathbf{U}^{(1,l)}(s))]_{j,k} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n}, \\ \Theta_{3,l} &= \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} \mathbf{U}^{(1,l)}(s)]_{j,k} [(\mathbf{U}(t-s) - \mathbf{U}^{(1,l)}(t-s)) \mathbf{H}^{(1)}]_{j,k+n}, \\ \Theta_{4,l} &= \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} \mathbf{U}^{(1,l)}(s)]_{j,k} [\mathbf{U}^{(1,l)}(t-s) (\mathbf{H}^{(1)} - \mathbf{H}^{(1,l)})]_{j,k+n}. \end{aligned}$$

It is easy to check that $\Theta_{1,l} = \Theta_{4,l} = 0$. We consider the term $\Theta_{2,l}$. The term $\Theta_{3,l}$ is similar. Applying (B.3) we get

$$\Theta_{2,l} = I_{1,l} + I_{2,l},$$

where

$$\begin{aligned} I_{1,l} &= \int_0^s \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} \mathbf{U}^{(1,l)}(s_1) \mathbf{e}_l \mathbf{e}_l^T \mathbf{V} \mathbf{J} \mathbf{U}(s-s_1)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n} ds_1, \\ I_{2,l} &= \int_0^s \sum_{j,k=1}^n [\tilde{\mathbf{H}}^{(2,l)} \mathbf{J} \mathbf{U}^{(1,l)}(s_1) \mathbf{V} \mathbf{J} \mathbf{e}_l \mathbf{e}_l^T \mathbf{U}(s-s_1)]_{k,j} [\mathbf{U}(t-s) \mathbf{H}^{(1)}]_{j,k+n} ds_1, \end{aligned}$$

Doing simple calculations we get

$$I_{1,l} = \int_0^s [\mathbf{W} \mathbf{U}_3(s-s_1) \mathbf{U}_2(t-s) (\mathbf{Y}^{(2)})^T \mathbf{Y}^{(2)} \mathbf{U}_3^{(1,l)}(s_1)]_{ll} ds_1,$$

It is easy to derive the following estimate

$$\sum_{l=1}^n \mathbb{E} I_{1,l}^2 \leq C s^2 \mathbb{E} \|\mathbf{W} (\mathbf{Y}^{(2)})^T \mathbf{Y}^{(2)}\|_2^2 \leq C s^2 n.$$

The same is true for $\sum_{l=1}^n \mathbb{E} I_{2,l}^2$. This fact finishes the proof of Lemma. \square

The following lemma gives an estimate for the variance of $\frac{1}{n} u_n(t)$, $\mathbf{V}_{n,j}(t)$ and $T_j(t) := [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t) \mathbf{H}^{(1)}]_{j,j+n}$ for all $j = 1, \dots, n$.

Lemma C.2. *Under condition of Theorem 1.1 we have*

$$(C.1) \quad \text{Var} \left[\frac{1}{n} u_n(t) \right] \leq \frac{C}{n},$$

$$(C.2) \quad \text{Var}[V_{n,j}(t)] = o(1) \quad j = 1, \dots, n;$$

$$(C.3) \quad \text{Var}[T_j(t)] = o(1) \quad j = 1, \dots, n.$$

Proof. The proof of the first statement (C.1) may be realized similarly to the proof of Lemma C.1. One may also use the result for the matrix resolvent and the Stieltjes transform. We present the proof of (C.3) only. The proof of (C.2) is similar. Let us denote

$$K_{j,n}(t_1, t_2) = \mathbb{E}[T_j(t_1)(T_j(t_2) - \mathbb{E} T_j(t_2))] = \mathbb{E} T_j(t_1) T_j^0(t_2),$$

where $T_j^0(t) := T_j(t) - \mathbb{E} T_j(t)$. We have

$$K_{j,n}(t_1, t_2) = \frac{1}{\sqrt{n}} \sum_{k=1}^n Y_{jk}^{(2)} [\mathbf{U}(t_1) \mathbf{H}^{(2)}]_{k+n, j+n} T_j^0(t_2)$$

By Taylor's formula

$$\begin{aligned} K_{j,n}(t_1, t_2) &= \frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E} \left[\frac{\partial \mathbf{U}(t_1)}{\partial Y_{jk}^{(2)}} \mathbf{H}^{(1)} \right]_{k+n, j+n} T_j^0(t_2) \\ &+ \frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E} \left[\mathbf{U}(t_1) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{k+n, j+n} T_j^0(t_2) \\ &+ \frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E} [\mathbf{U}(t_1) \mathbf{H}^{(1)}]_{k+n, j+n} \left[\frac{\partial \mathbf{H}^{(2)}}{\partial Y_{jk}^{(2)}} \mathbf{J} \mathbf{U}(t_2) \mathbf{H}^{(1)} \right]_{j, j+n} \\ &+ \frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E} [\mathbf{U}(t_1) \mathbf{H}^{(1)}]_{k+n, j+n} \left[\mathbf{H}^{(2)} \mathbf{J} \frac{\partial \mathbf{U}(t_2)}{\partial Y_{jk}^{(2)}} \mathbf{H}^{(1)} \right]_{j, j+n} \\ &+ \frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E} [\mathbf{U}(t_1) \mathbf{H}^{(1)}]_{k+n, j+n} \left[\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t_2) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}} \right]_{j, j+n} \\ &+ R, \end{aligned}$$

where R is a remainder term. It is straightforward to check R has the order $o(1)$. By Lemma B.3 we get

$$\begin{aligned}
K_{j,n}(t_1, t_2) &= \frac{i}{n} \mathbb{E} u_n * [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t_1) \mathbf{H}^{(1)}]_{j+n, j+n} T_j^0(t_2) \\
&+ \frac{i}{n} \sum_{k=1}^n \mathbb{E} [\mathbf{U} \mathbf{H}^{(1)}]_{k+n, j+n} * [\mathbf{U}(t_1) \mathbf{H}^{(1)}]_{k+n, j+n} T_j^0(t_2) \\
&+ \frac{1}{n} \mathbb{E} u_n(t_1) T_j^0(t_2) \\
&+ \frac{2}{n} \sum_{k=1}^n \mathbb{E} [\mathbf{U}(t_1) \mathbf{H}^{(1)}]_{k+n, j+n} [\mathbf{U}(t_2) \mathbf{H}^{(1)}]_{k+n, j+n} \\
&+ \frac{2i}{n} \sum_{k=1}^n \mathbb{E} [\mathbf{U}(t_1) \mathbf{H}^{(1)}]_{k+n, j+n} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}]_{j, k+n} * [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t_2) \mathbf{H}^{(1)}]_{j+n, j+n} \\
&+ R.
\end{aligned}$$

Similarly to the previous estimates it is not very difficult to check that all term except the first one have the order $o(1)$. Let us consider the first term

$$\begin{aligned}
&\frac{i}{n} \mathbb{E} u_n * [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t_1) \mathbf{H}^{(1)}]_{j+n, j+n} T_j^0(t_2) \\
&= \frac{i}{n} \mathbb{E} u_n * \mathbb{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t_1) \mathbf{H}^{(1)}]_{j+n, j+n} T_j^0(t_2) + o(1).
\end{aligned}$$

From (3.9) we have

$$V_{n,k}(s) = \frac{i}{n} \mathbb{E} u_n(s) * \mathbb{E} T_k(s) + o(1).$$

We may conclude that

$$\begin{aligned}
&\frac{i}{n} \sum_{k=1}^n \mathbb{E} [\mathbf{U}]_{k+n, k+n} * \mathbb{E} [\mathbf{H}^{(2)} \mathbf{J} \mathbf{U}(t_1) \mathbf{H}^{(1)}]_{j+n, j+n} T_j^0(t_2) \\
&= -\frac{1}{n^2} (\mathbb{E} u_n)^{*2} * \mathbb{E} T_j(t_1) T_j^0(t_2) + o(1).
\end{aligned}$$

Taking the limit with respect to $n_l \rightarrow \infty$ we get that $K_j := \lim_{n_l \rightarrow \infty} K_{j, n_l}$ satisfies the following equation

$$K_j(t_1, t_2) = - \int_0^{t_1} v^{*2}(t_1 - s) K_j(s, t_2) ds.$$

Since $K_j(t_1, t_2) = 0$ is a unique solution of the last equation this means that

$$K_{j,n}(t_1, t_2) = o(1).$$

One may take $t_2 = t_1$ and finish the proof of Lemma. \square

APPENDIX D. LAPLACE TRANSFORM

In this section we recall several results from the theory of Laplace transform. We will follow [19][Proposition 2.1].

Statement D.1. Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be locally Lipschitzian and such that for some $\delta > 0$

$$\sup_{t \geq 0} e^{-\delta t} |f(t)| < \infty$$

and let $\tilde{f} : \{z \in \mathbb{C} : \operatorname{Im} z < -\delta\} \rightarrow \mathbb{C}$ be its generalized Fourier transform

$$\tilde{f}(z) = \frac{1}{i} \int_0^\infty e^{-izt} f(t) dt.$$

The inversion formula is given by

$$f(t) = \frac{i}{2\pi} \int_L e^{izt} \tilde{f}(z) dz, \quad t \geq 0,$$

where $L = (-\infty - i\varepsilon, \infty - i\varepsilon)$, $\varepsilon > \delta$, and the principal value of the integral at infinity is used. Denote the correspondence between functions and their generalized Fourier transforms as $f \leftrightarrow \tilde{f}$. Then we have

$$\begin{aligned} f'(t) &\leftrightarrow i(f(+0) + z\tilde{f}(z)); \\ \int_0^t f(s) ds &\leftrightarrow (iz)^{-1} \tilde{f}(z); \\ f * g(t) &\leftrightarrow i\tilde{f}(z)\tilde{g}(z). \end{aligned}$$

We calculate the Fourier transforms of some functions.

Lemma D.2. Let $s(z)$ be the Stieltjes transform of $p(x)$ which is a symmetrization of Fuss-Catalan density $P_2(x)$, see Appendix A. The inverse Fourier transform of

$$K(z) = \frac{1/z - 2s(z)}{1 - 3s^2(z)}$$

is given by

$$T(t) = \frac{1}{\pi} \int_{-a}^a \frac{e^{itx}}{3p_1(x)} \frac{4p_1^4(x) + 11p_1^2(x) + 4}{4p_1^2(x) + 3} dx,$$

where $p_1(x) = \pi p(x)$.

Proof. Be definition, see Statement D.1,

$$T(t) = \frac{i}{2\pi} \int_L e^{itz} K(z) dz,$$

where $L = (-\infty - i\varepsilon, \infty - i\varepsilon)$. We introduce the following contour \mathbf{K} (see Figure 1) :

$$\mathbf{K} := \mathbf{K}_1 \cup \dots \cup \mathbf{K}_8,$$

where

$$\begin{aligned} \mathbf{K}_1 &:= \{z = u + iv, |u| \leq T, v = -\varepsilon\}, & \mathbf{K}_2 &:= \{z = u + iv : |z| = T, v \geq 0\}, \\ \mathbf{K}_{3,4} &:= \{z = u + iv : |u| \leq a + \varepsilon/2, v = \pm\varepsilon/2\}, \\ \mathbf{K}_{5,6} &:= \{z = u + iv : u = \pm(a + \varepsilon/2), -\varepsilon/2 \leq v \leq \varepsilon/2\}, \\ \mathbf{K}_{7,8} &:= \{z = u + iv : u = \pm T, -\varepsilon \leq v \leq 0\}. \end{aligned}$$

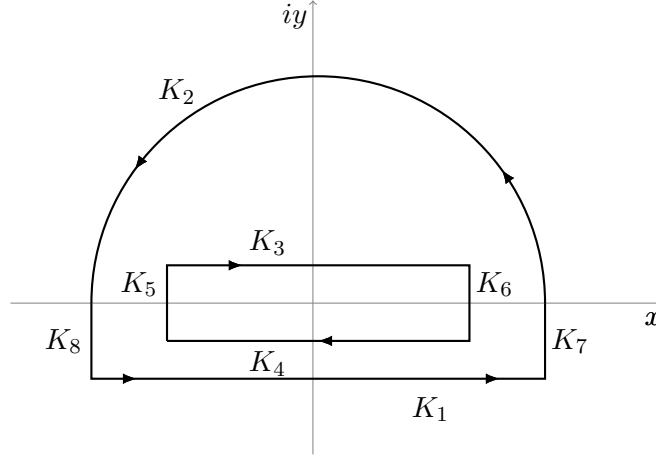


FIGURE 1. Contour of integration

We may write

$$T(t) = \lim_{T \rightarrow \infty} \frac{i}{2\pi} \int_{\mathbf{K}_1} e^{itz} K(z) dz$$

and

$$\int_{\mathbf{K}} e^{itz} K(z) dz = 0.$$

Furthermore, we note

$$\lim_{T \rightarrow \infty} \int_{\mathbf{K}_2 \cup \mathbf{K}_7 \cup \mathbf{K}_8} e^{itz} K(z) dz = 0.$$

We compute the integrals

$$\mathcal{K}_1 := \left(\int_{\mathbf{K}_3} - \int_{\mathbf{K}_4} \right) e^{itz} K(z) dz.$$

Let $s(z) = if(z) + g(z)$, for $z = u + iv$ with $v > 0$. Note that by definition

$$\operatorname{Im} s(z) = \begin{cases} f(z), & \text{if } \operatorname{Im} z > 0, \\ -f(z), & \text{if } \operatorname{Im} z < 0. \end{cases}$$

Let us calculate $K(z)$ for $z \in K_3$. Applying

$$1 + zs(z) = zs^3(z)$$

we get

$$\begin{aligned} K(z) &= \frac{s(z)(s^2(z) - 3)}{1 - 3s^2(z)} = \frac{1}{3} \frac{(g + if)(f^2 + 1 - 3f^2 + 6ifg - 9)}{2p(f - 3ig)} \\ &= \frac{1}{6f} \frac{(g + if)(6ifg - 2f^2 - 8)}{f - 3ig} = \frac{1}{6f} \frac{(g + if)(6ifg - 2f^2 - 8)(f + 3ig)}{|f - 3ig|^2}. \end{aligned}$$

The enumerator is equal to

$$\begin{aligned} (g + if)(6ifg - 2f^2 - 8)(f + 3ig) &= (6ifg^2 - 2f^2g - 8g - 6f^2g - 2if^3 - 8if)(f + 3ig) \\ &= (2if + 2if^3 - 8f^2g - 8g - 2if^3 - 8if)(f + 3ig) \\ &= -2(3if + 4f^2g + 4g)(f + 3ig). \end{aligned}$$

The imaginary part of the enumerator is given by

$$-6f^2 - 24f^2g^2 - 24g^2 = -6f^2 - 8f^2 - 8f^4 - 8 - 8f^2 = -2(4f^4 + 11f^2 + 4).$$

Finally

$$\operatorname{Im} K(z) = -\frac{1}{3f} \frac{4f^4 + 11f^2 + 4}{4f^2 + 3}.$$

The real part is equal to

$$\operatorname{Re} K(z) = -\frac{1}{3} \frac{g(5 - 4f^2)}{4f^2 + 3}.$$

It is easy to see that for $z \in K_4$ we will have

$$\operatorname{Im} K(z) = \frac{1}{3f} \frac{4f^4 + 11f^2 + 4}{4f^2 + 3}, \operatorname{Re} K(z) = -\frac{1}{3} \frac{g(5 - 4f^2)}{4f^2 + 3}.$$

Since ε is an arbitrary number and

$$\lim_{\varepsilon \rightarrow 0} f(u + i\varepsilon) = \pi p(u)$$

then integrating $\operatorname{Re} K(z)$ in the opposite directions we get zero. Finally

$$T(t) = \frac{1}{\pi} \int_{-a}^a \frac{e^{itx}}{3\pi p(x)} \frac{4(\pi p(x))^4 + 11(\pi p(x))^2 + 4}{4(\pi p(x))^2 + 3} dx.$$

□

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F. GÖTZE, FACULTY OF MATHEMATICS, BIELEFELD UNIVERSITY, BIELEFELD, GERMANY

E-mail address: `goetze@math.uni-bielefeld.de`

A. NAUMOV, FACULTY OF COMPUTATIONAL MATHEMATICS AND CYBERNETICS, MOSCOW STATE UNIVERSITY, MOSCOW, RUSSIA

E-mail address: `anaumov@cs.msu.su`

A. TIKHOMIROV, DEPARTMENT OF MATHEMATICS, KOMI RESEARCH CENTER OF URAL DIVISION OF RAS, SYKTYVKAR, RUSSIA

E-mail address: `tichomir@math.uni-bielefeld.de`